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Games and Natural Number-valued Semantics of the Modal μ-calculus

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The modal μ-calculus has strong expressive power to describe properties of Kripke structures. The semantics of the logic can be expressed using games: for a given Kripke structure, a parity game between Player and Opponent can be defined so that a state satisfies a formula if and only if the corresponding vertex of the game belongs to the winning region of Player. The games play an important role in the standard decision procedure for satisfiability. In previous research, we introduced a non-standard semantics of the logic: a truth value of a formula is an element of a natural number or the infinity. Thus, formulae express quantitative properties of Kripke structures. In this work, toward the research of the satisfiability problem of our semantics, we develop a technique which is a counterpart of the game expression of the ordinary semantics.

1 Introduction

Modal μ-calculus [5], like its various sublogics, has widely been used as a language to describe properties of Kripke structures. Despite of its strong expressibility, the satisfiability problem of the logic is decidable [2]. This fact leads to applications of the logic such as synthesizing programs from temporal properties [1] [6] or verification of heap manipulating programs [8].

The authors introduced a non-standard semantics of the logic [4]. The semantics is given over Kripke structures as the ordinary one, but the truth values of formulae are elements of $\mathbf{N}_\infty = \mathbf{N} \cup \{\infty\}$. Thus, formulae can express quantitative properties of Kripke structures such as the distance between two states, or the sum of the weight given to each status.

From application point of view, it is important to establish a model-checking algorithm and a decision procedure for satisfiability. Unlike the ordinary semantics, even existence of a procedure for model-checking is non-trivial, as the naive repetition method may not terminate because the reverse magnitude relation $>$ on $\mathbf{N}_\infty$ is not well-founded. However, using a suitable acceleration technique, we can build a model-checking algorithm [7].

For the satisfiability problem, we have a partial result. Let us denote the truth value of formula φ at state s of a Kripke structure $\mathcal{K}$ by $[\![\varphi]\!]^{\mathcal{K}}(s)$. We built a procedure that decides whether there exist $\mathcal{K}$ and s such that $[\![\varphi]\!]^{\mathcal{K}}(s) = 0$ holds. The procedure first translates a given formula φ to φ' that is equi-satisfiable to φ, namely, there is $\mathcal{K}$ and s such that $[\![\varphi]\!]^{\mathcal{K}}(s) = 0$ under the $\mathbf{N}_\infty$ semantics if and only if φ' is satisfiable under the ordinary semantics. Thus, by combined with a decision procedures for the ordinary semantics, the translation gives us a decision procedure for the $\mathbf{N}_\infty$ semantics.

A similar translation is possible even if we replace 0 with ∞. We naturally wish to expand the result for any $n \in \mathbf{N}_\infty$. However, a small experiment shows that defining a translation for arbitrary $n \in \mathbf{N}_\infty$ is too complicated. Rather, we apparently have more practical chance to expand a decision procedure for the ordinary semantics directly so that it fits the $\mathbf{N}_\infty$ semantics. The standard decision procedure for the ordinary semantics depends on the fact that for given Kripke structure $\mathcal{K}$ and formula φ, a parity game $\mathcal{G}$ between Player and Opponent can be defined so that φ is satisfied

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at a state s of $\mathcal{K}$ if and only if the corresponding vertex of $\mathcal{G}$ to s belongs to the winning region of Player [3]. For expanding the decision procedure to the $\mathbf{N}_\infty$ semantics, we need to build a corresponding game for the semantics, which is the target of this paper.

For each subformula ψ of given formula φ, $n \in \mathbf{N}_\infty$, and state s of given Kripke structure $\mathcal{K}$, there is a corresponding node (ψ, s, n). Roughly, Player plays to show $[\![\psi]\!]^{\mathcal{K}}(s) < n+1$ and Opponent plays to rebut it. We give a precise definition of the game and show that $[\![\varphi]\!]^{\mathcal{K}}(s) < n+1$ if and only if (φ, s, n) belongs to the winning region of Player. The game is carefully designed so that if the initial vertex is (φ, s, n), then any visited vertex (ψ, t, m) satisfies $m \in \{\infty, 0, 1, \ldots, n\}$. Thus, if $\mathcal{K}$ is finite, then the set of necessary vertices is also finite. This property will be crucial to build a decision procedure for satisfiability using the game.

The rest of the paper is organized as follows. In Section 2, we review the syntax of the logic and the $\mathbf{N}_\infty$ semantics. We also introduce several concepts and lemmas that are needed to describe strategies. In Section 3, we define the game and strategies for Player and Opponent. Then, the equivalence proof is given in Section 4. Section 5 concludes the paper.

2 Preliminaries

2.1 Syntax and Semantics

Let PS be the set of propositional symbols and PV be the set of propositional variables. The formulae of the modal μ-calculus are defined as follows:

$$\varphi ::= p \mid X \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu X\varphi \mid \nu X\varphi$$

where $p \in \mathrm{PS}$ and $X \in \mathrm{PV}$. All occurrences of X in $\mu X\varphi$ and $\nu X\varphi$ must be positive in φ. That is, the number of negations of which the occurrence is in the scope must be even.

$\mathcal{K} = (S, R, L)$ is a *Kripke structure* if S is a set, $R \subseteq S \times S$, and $L : \mathrm{PS} \times S \to \mathbf{N}_\infty$. In this paper, S can be finite or infinite. A function $\rho : \mathrm{PV} \times S \to \mathbf{N}_\infty$ is called a *valuation*. For formula φ and $t \in T$, the value $[\![\varphi]\!]^{\mathcal{K},\rho}(s) \in \mathbf{N}_\infty$ of φ at s is defined as in Figure 1. $\mathcal{K}$ and/or ρ are omitted if they are clear from the context. In Figure 1, sR is the set $\{s' \in S \mid (s, s') \in R\}$ for $s \in S$. On is the class of ordinal numbers. For function f, $f[a \mapsto b]$ is the function g whose domain is $\mathrm{dom}(f) \cup \{a\}$, and whose values are defined by $g(a) = b$ and $g(x) = f(x)$ for any $x \in \mathrm{dom}(f) \setminus \{a\}$.

$$[\![p]\!]^\rho(s) = L(p, s)$$

$$[\![X]\!]^\rho(s) = \rho(X, s)$$

$$[\![\neg\psi]\!]^\rho(s) = \begin{cases} 0 & \text{if } [\![\psi]\!]^\rho(s) = \infty \\ \infty & \text{if } [\![\psi]\!]^\rho(s) < \infty \end{cases}$$

$$[\![\psi_1 \vee \psi_2]\!]^\rho(s) = \min([\![\psi_1]\!]^\rho(s), [\![\psi_2]\!]^\rho(s))$$

$$[\![\psi_1 \wedge \psi_2]\!]^\rho(s) = [\![\psi_1]\!]^\rho(s) + [\![\psi_2]\!]^\rho(s)$$

$$[\![\Diamond\psi]\!]^\rho(s) = \min([\![\psi]\!]^\rho(s') \mid s' \in sR)$$

$$[\![\Box\psi]\!]^\rho(s) = \textstyle\sum([\![\psi]\!]^\rho(s') \mid s' \in sR)$$

$$[\![\mu X\psi]\!]^\rho(s) = \inf\{F_\alpha(s) \mid \alpha \in \mathrm{On}\}, \quad \text{where } F_\alpha(s') = \inf\{[\![\psi]\!]^{\rho[X \mapsto F_\beta]}(s') \mid \beta < \alpha\}$$

$$[\![\nu X\psi]\!]^\rho(s) = \sup\{F_\alpha(s) \mid \alpha \in \mathrm{On}\}, \quad \text{where } F_\alpha(s') = \sup\{[\![\psi]\!]^{\rho[X \mapsto F_\beta]}(s') \mid \beta < \alpha\}$$

Fig. 1 Values of formulae

Note that the distributive law $[\![\varphi \vee (\psi_1 \wedge \psi_2)]\!](s) = [\![(\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2)]\!](s)$ does not hold. De Morgan's law is not satisfied: $[\![\neg\Box\varphi]\!](s) = [\![\Diamond\neg\varphi]\!](s)$ does not necessarily hold if state s has infinite successors. Double negation cannot be eliminated: $[\![\neg\neg\varphi]\!](s) \neq [\![\varphi]\!](s)$ unless $[\![\varphi]\!](s) = 0$ or $[\![\varphi]\!](s) = \infty$.

Let φ_{I} be a given closed formula, and $\mathcal{K} = (S, R, L)$ is a given Kripke structure. We only consider propositional variables occurring in φ_{I}. Thus, PV is the set of propositional variables that occur in φ_{I}. Without loss of generality, we assume that each propositional variable X is bound in φ_{I} only once. Symbol σ is used to express either operator μ or operator ν. The *bounding formula* of X is denoted by $\mathrm{BF}(X)$: i.e., if $\sigma X\psi$ occurs in φ_{I}, then $\mathrm{BF}(X) = \sigma X\psi$. The direct subformula of $\mathrm{BF}(X)$, ψ in this case, is denoted by $\mathrm{BFS}(X)$. The corresponding fixed-point operator is denoted by σ_X. For example, if $\mu X\nu Y\varphi \in \mathrm{SF}$, then $\sigma_X = \mu$ and $\sigma_Y = \nu$.

Unless explicitly stated otherwise, a "subformula" means its occurrence. For example, formula $\varphi = X \wedge \neg X$ has four subformulas: φ itself, $\neg X$, and two occurrences of X. We write $\psi \leq \varphi$ if ψ is a subformula of φ. The set of subformulas of φ_{I} is denoted by SF. The set SF can be divided into two: the set SF^+ of *positive* subformulas and the set SF^- of *negative* subformulas. They are defined as follows:

- $\varphi_{\mathrm{I}} \in \mathrm{SF}^+$
- If $\neg\varphi \in \mathrm{SF}^+$ (SF^- resp.), then $\varphi \in \mathrm{SF}^-$ (SF^+ resp.).
- If ψ is a direct subformula of $\varphi \in \mathrm{SF}^+$ (SF^- resp.), then $\psi \in \mathrm{SF}^+$ (SF^- resp.).

If $\varphi \in \mathrm{SF}^+$, we write $\mathrm{sgn}(\varphi) = 1$, otherwise, $\mathrm{sgn}(\varphi) = -1$. For any $X \in \mathrm{PV}$, $\mathrm{sgn}(X) = \mathrm{sgn}(\mathrm{BF}(X)) = \mathrm{sgn}(\mathrm{BFS}(X))$.

For each $\varphi \in \mathrm{SF}$, we define the set $\mathrm{FV}(\varphi)$ of (possible) *free variables* in φ by $\mathrm{FV}(\varphi) = \{X \in \mathrm{PV} \mid \varphi < \mathrm{BF}(X)\}$. We define a partial order $\prec$ on PV: $X \preceq Y$ if and only if $\mathrm{BF}(X) \leq \mathrm{BF}(Y)$. The *index* $\mathrm{idx}(X)$ of $X \in \mathrm{PV}$ is the number of propositional variables Y such that $X \prec Y$.

2.2 Intermediate Values

Let Seq be the class of sequences of ordinal numbers. $\mathrm{Seq}_l = \{\xi \in \mathrm{Seq} \mid \mathrm{len}(\xi) = l\}$ for $l \in \mathbf{N}$, and $\mathrm{Seq}_\varphi = \mathrm{Seq}_{|\mathrm{FV}(\varphi)|}$ for $\varphi \in \mathrm{SF}$. An ordinal number α is considered as a sequence of length one, i.e., an element of Seq_1. For $\xi \in \mathrm{Seq}$, its l-th element is denoted by $\xi(l)$. For $\xi, \xi' \in \mathrm{Seq}$, their concatenation is denoted by $\xi : \xi'$. If $n \geq \mathrm{len}(\xi)$, the prefix of ξ whose length is n is denoted by $\xi \restriction n$.

We define a well order on $\mathrm{Seq}_{\varphi_{\mathrm{I}}}$: $\xi \leq \eta$ holds if either there is l such that $\xi \restriction l = \eta \restriction l$ and $\xi(l) < \eta(l)$ or ξ is an extension of η, i.e., $\eta = \xi \restriction \mathrm{len}(\eta)$. It is similar to the lexicographical order, but the order on the extension is reversed. This is a well order on $\bigcup_{\varphi \in \mathrm{SF}} \mathrm{Seq}_\varphi$.

Let $\varphi \in \mathrm{SF}$, $\xi \in \mathrm{Seq}_\varphi$, and $s \in S$. We define *intermediate values* $\mathrm{iv}(\varphi, \xi, s)$ as in Figure 2.

The relation between $[\![\cdot]\!]$ and $\mathrm{iv}(\cdot,\cdot,\cdot)$ is as follows. For $\xi \in \mathrm{Seq}$, we define valuation ρ_ξ by $\rho_\xi(X)(s) = \mathrm{iv}(X, \xi, s)$. (The right hand side is not defined if $\mathrm{idx}(X) > \mathrm{len}(\xi)$. In such cases, we assign arbitrary values, 0 for example.)

Lemma 1. *For any $\varphi \in \mathrm{SF}$ and $\xi \in \mathrm{Seq}_\varphi$, $\mathrm{iv}(\varphi, \xi, s) = [\![\varphi]\!]^{\rho_\xi}(s)$. In particular, $\mathrm{iv}(\varphi_{\mathrm{I}}, \epsilon, s) = [\![\varphi_{\mathrm{I}}]\!](s)$, where ϵ is the empty sequence.*

Proof. By a straight-forward induction on the construction of φ. □

Lemma 2. *There is an ordinal number κ such that $\mathrm{iv}(\sigma X\varphi,\ \xi,\ s) = \mathrm{iv}(\varphi,\ \xi : \kappa,\ s)$ for all $\sigma X\varphi \in \mathrm{SF}$, $\xi \in \mathrm{Seq}_{\sigma X\varphi}$, and $s \in S$.*

Proof. This can be achieved by taking κ sufficiently large. For example, it is sufficient to take κ as an uncountable cardinal number bigger than the cardinality of S. □

$$\mathrm{iv}(p, \xi, s) = L(p)$$

If $\sigma_X = \mu$, then $\mathrm{iv}(X, \xi, s) =$

$$\begin{cases} \infty & \text{if } \xi(l) = 0 \\ \mathrm{iv}(\varphi, \xi' : \alpha, s) & \text{if } \xi(l) = \alpha + 1 \\ \inf_{\beta < \xi(l)} \mathrm{iv}(\varphi, \xi' : \beta, s) & \text{if } \xi(l) \text{ is limit.} \end{cases}$$

If $\sigma_X = \nu$, then $\mathrm{iv}(X, \xi, s) =$

$$\begin{cases} 0 & \text{if } \xi(l) = 0 \\ \mathrm{iv}(\varphi, \xi' : \alpha, s) & \text{if } \xi(l) = \alpha + 1 \\ \sup_{\beta < \xi(l)} \mathrm{iv}(\varphi, \xi' : \beta, s) & \text{if } \xi(l) \text{ is limit.} \end{cases}$$

where $\varphi = \mathrm{BFS}(X)$, $l = \mathrm{idx}(X)$, and $\xi' = \xi \restriction l$.

$$\mathrm{iv}(\neg\varphi, \xi, s) = \begin{cases} 0 & \text{if } \mathrm{iv}(\varphi, \xi, s) = \infty \\ \infty & \text{if } \mathrm{iv}(\varphi, \xi, s) < \infty \end{cases}$$

$$\mathrm{iv}(\varphi_1 \vee \varphi_2, \xi, s) = \min(\mathrm{iv}(\varphi_1, \xi, s), \mathrm{iv}(\varphi_2, \xi, s))$$

$$\mathrm{iv}(\varphi_1 \wedge \varphi_2, \xi, s) = \mathrm{iv}(\varphi_1, \xi, s) + \mathrm{iv}(\varphi_2, \xi, s)$$

$$\mathrm{iv}(\Diamond\varphi, \xi, s) = \min(\mathrm{iv}(\varphi, \xi, t) \mid t \in sR)$$

$$\mathrm{iv}(\Box\varphi, \xi, s) = \textstyle\sum(\mathrm{iv}(\varphi, \xi, t) \mid t \in sR)$$

$$\mathrm{iv}(\mu X\varphi, \xi, s) = \inf\{\mathrm{iv}(\varphi, \xi : \alpha, s) \mid \alpha \in \mathrm{On}\}$$

$$\mathrm{iv}(\nu X\varphi, \xi, s) = \sup\{\mathrm{iv}(\varphi, \xi : \alpha, s) \mid \alpha \in \mathrm{On}\}$$

Fig. 2 Intermediate values

We fix an ordinal number κ that satisfies the condition in Lemma 2.

The value of $\mathrm{iv}(\varphi, \xi, s)$ increases or decreases when the value of ξ changes. For example, assume we have $\xi, \eta \in \mathrm{Seq}_\varphi$, and $\xi(l) < \eta(l)$ and $\xi(i) = \eta(i)$ for all i except for l. Which of $\mathrm{iv}(\varphi, \xi, s)$ and $\mathrm{iv}(\varphi, \eta, s)$ is bigger depends on the corresponding variable X that satisfies $\mathrm{idx}(X) = l$. There are two factors: whether $\mathrm{sgn}(\varphi) = \mathrm{sgn}(X)$ and whether $\sigma_X = \nu$. If none or both of them are satisfied, then $\mathrm{iv}(\varphi, \xi, s) \leq \mathrm{iv}(\varphi, \eta, s)$, otherwise $\mathrm{iv}(\varphi, \xi, s) \geq \mathrm{iv}(\varphi, \eta, s)$.

Based on this observation, we introduce the following definitions. Let $\varphi \in \mathrm{SF}$. We divide $\mathrm{FV}(\varphi)$ into two subsets, $\mathrm{FV}_{\mathrm{I}}(\varphi)$ and $\mathrm{FV}_{\mathrm{D}}(\varphi)$. If either $\mathrm{sgn}(\varphi) = \mathrm{sgn}(X)$ and $\sigma_X = \nu$ or $\mathrm{sgn}(\varphi) \neq \mathrm{sgn}(X)$ and $\sigma_X = \mu$, then $X \in \mathrm{FV}_{\mathrm{I}}(\varphi)$. Otherwise, $X \in \mathrm{FV}_{\mathrm{D}}(\varphi)$. Letters 'I' and 'D' stand for "increasing" and "decreasing", respectively, Then two subsets $\mathrm{Seq}_{\mathrm{I}}(\varphi)$ and $\mathrm{Seq}_{\mathrm{D}}(\varphi)$ of $\mathrm{Seq}(\varphi)$ are defined by $\xi \in \mathrm{Seq_I}(\varphi)$ if and only if $\xi(\mathrm{idx}(X)) = \kappa$ for all $X \in \mathrm{FV_D}(\varphi)$, and $\xi \in \mathrm{Seq_D}(\varphi)$ if and only if $\xi(\mathrm{idx}(X)) = \kappa$ for all $X \in \mathrm{FV_I}(\varphi)$. The two sets behave well, as shown in the next lemma.

Lemma 3. *Assume $s \in S$. If $\xi, \eta \in \mathrm{Seq_I}(\varphi)$ and $\xi < \eta$, then $\mathrm{iv}(\varphi, \xi, s) \le \mathrm{iv}(\varphi, \eta, s)$. If $\xi, \eta \in \mathrm{Seq_D}(\varphi)$ and $\xi < \eta$, then $\mathrm{iv}(\varphi, \xi, s) \ge \mathrm{iv}(\varphi, \eta, s)$.*

Proof. By easy induction on φ. □

2.3 Singular Variables

If there exists $\xi \in \mathrm{Seq_I}(\varphi)$ such that $\mathrm{iv}(\varphi, \xi, s) \ge n+1$, we denote the least such ξ by $\mathrm{lseq_I}(\varphi, s, n)$, otherwise $\mathrm{lseq_I}(\varphi, s, n)$ is undefined. If there exists $\xi \in \mathrm{Seq_D}(\varphi)$ such that $\mathrm{iv}(\varphi, \xi, s) < n+1$, we denote the least such ξ by $\mathrm{lseq_D}(\varphi, s, n)$, otherwise $\mathrm{lseq_D}(\varphi, s, n)$ is undefined.

A propositional variable $X \in \mathrm{FV_I}(\varphi)$ is *singular* in φ and s if the following conditions are satisfied.

- $\sigma_X = \nu$. (Hence, $\mathrm{sgn}(\varphi) = \mathrm{sgn}(X)$.)
- $\xi = \mathrm{lseq_I}(\varphi, s, \infty)$ is defined.
- $\xi(l)$ is a limit ordinal, where $l = \mathrm{idx}(X)$.
- For all $k \in \mathbf{N}$, there exists $\eta_k \in \mathrm{Seq_I}(\varphi)$ such that $\eta_k \upharpoonright (l+1) < \xi \upharpoonright (l+1)$ and $\mathrm{iv}(\varphi, \eta_k, s) \ge k$.

Lemma 4. *Assume that X is singular in φ and s, $l = \mathrm{idx}(X)$, and $\xi = \mathrm{lseq_I}(\varphi, s, \infty)$.*

(1) If $\varphi = Y \in \mathrm{PV}$ and $Y \preceq X$, then X is singular in $\mathrm{BFS}(Y)$ and s.

(2) φ is not in the form of $\neg\psi$.

(3) If $\varphi = \psi_1 \vee \psi_2$, then for each of $j = 1$ and $j = 2$, either X is singular in ψ_j and s, or $\mathrm{lseq_I}(\psi_j, s, \infty) \upharpoonright (l+1) < \xi \upharpoonright (l+1)$.

(4) If $\varphi = \psi_1 \wedge \psi_2$, then X is singular in ψ_j and s for either $j = 1$ or $j = 2$.

(5) If $\varphi = \Diamond\psi$, then for each $t \in sR$, either X is singular in ψ and t, or $\mathrm{lseq_I}(\psi, t, \infty) \upharpoonright (l+1) < \xi \upharpoonright (l+1)$.

(6) If $\varphi = \Box\psi$, then either there exists $t \in sR$ such that X is singular in ψ and t, or there exist infinitely many $t \in sR$ such that $\xi_t = \mathrm{lseq_I}(\psi, t, 0)$ is defined and $\xi_t \upharpoonright (l+1) < \xi \upharpoonright (l+1)$.

(7) If $\varphi = \sigma Y \psi$ for some $Y \preceq X$, then X is singular in ψ and s.

Proof.

(1) By the assumption, we have $\mathrm{iv}(Y, \xi, s) = \infty$ and $\mathrm{iv}(Y, \eta_k, s) \ge k$. Let $m = \mathrm{idx}(Y)$.

If $\sigma_Y = \mu$, then $m \in \mathrm{FV_D}(\varphi)$ and $\mathrm{lseq_D}(\mathrm{BFS}(Y)) = \xi \upharpoonright m$. It is easy to check that X is singular for $\mathrm{BFS}(Y)$ and s, using $\eta_k \upharpoonright (m+1)$.

If $\sigma_Y = \nu$, then $\mathrm{lseq_D}(\mathrm{BFS}(Y)) = \xi' : \alpha$, where $\xi' = \xi \upharpoonright m$ and either $\xi(m) = \alpha + 1$, or $\xi(m) = \alpha$ and α is a limit ordinal. To show that $\mathrm{BFS}(Y)$ is singular, we can use $\eta_k \upharpoonright m : \beta_k$, where either $\eta_k(m) = \beta_k + 1$, or $\eta_k(m) = \beta_k$ and β_k is a limit ordinal.

(2) We have $1 \ge \mathrm{iv}(\varphi, \eta_1, s) < \infty$ by leastness of ξ. However, if $\varphi = \neg\psi$, the value of $\mathrm{iv}(\varphi, \eta_1, s)$ must be 0 or ∞.

(3) It is clear that $\mathrm{iv}(\psi_j, \xi, s) = \infty$ and $\mathrm{iv}(\psi_j, \eta_k, s) \ge k$ for $j = 1, 2$. therefore, if $\mathrm{lseq_I}(\psi_j, s, \infty) \upharpoonright (l+1) = \xi \upharpoonright (l+1)$, then η_k can be used as evidences. Otherwise, the latter half of the conclusion is satisfied.

(4) Let $V_j = \{\mathrm{iv}(\psi_j, \eta_k, s) \mid k \in \mathbf{N}\}$ for $j = 1, 2$. Because $V_1 \cup V_2$ is unbounded, either V_1 is unbounded or V_2 is unbounded. Assume V_j is unbounded. It is easy to show that $\xi = \mathrm{lseq_I}(\psi_j)$ and η_k satisfies the necessary condition.

(5) This case can be shown by a similar argument to (3).

(6) Let $V_k = \{t \in SR \mid \mathrm{iv}(\psi, \eta_k, t) \ge 1\}$. If $\bigcup_{k \in \mathbf{N}} V_k$ is an infinite set, the second condition of the conclusion is satisfied by taking $\xi_t = \eta_k$, where $t \in V_k$. Otherwise, there is $t \in \bigcup_{k \in \mathbf{N}} V_k$ such that $\{\mathrm{iv}(\psi, \eta_k, t) \mid k \in \mathbf{N}\}$ is unbounded. X is singular in ψ and t for this t.

(7) There is α and β_k for each $k \in \mathbf{N}$ such that $\mathrm{lseq_I}(\psi) = \xi : \alpha$ and $\mathrm{iv}(\varphi, \eta_k, s) = \mathrm{iv}(\psi, \eta_k : \beta_k, s)$. It is easy to check that X is singular using these sequences. □

3 Games

3.1 Moves and Priorities

We define a parity game $\mathcal{G}$ played by Player and Opponent. This game is used to decide whether $[\![\varphi_\mathrm{I}]\!](s_\mathrm{I}) < n_\mathrm{I} + 1$, for given $s_\mathrm{I} \in S$ and $n_\mathrm{I} \in \mathbf{N}_\infty$, which we fix from now on. A vertex of the game is in the form of (φ, s, n), where $\varphi \in \mathrm{SF}$, $s \in S$, and $n \in \{0, 1, \ldots, n_\mathrm{I}, \infty\}$. Thus, the set of vertices is finite if S is finite. However, in this paper. we do not assume that S is finite. Let us define $\gamma(\varphi, n) \subseteq \mathbf{N}_\infty$ by $\gamma(\varphi, n) = \{n' \in \mathbf{N}_\infty \mid n' < n+1\}$ if $\varphi \in \mathrm{SF}^+$ and $\gamma(\varphi, n) = \{n' \in \mathbf{N}_\infty \mid n' \ge n+1\}$ if $\varphi \in \mathrm{SF}^-$.

Table 1 Moves at vertex (φ, s, n)

φ	Side conditions	Turn	Moves
$\neg\psi$	—	(either)	(ψ, s, ∞)
$\psi_1 \vee \psi_2$	—	Decreaser	(ψ_1, s, n) or (ψ_2, s, n)
$\psi_1 \wedge \psi_2$	$n < \infty$	(1) Decreaser	$m_1, m_2 \in \mathbf{N}_\infty$ such that $m_1 + m_2 = n$.
		(2) Increaser	(ψ_1, s, m_1) or (ψ_2, s, m_2)
$\psi_1 \wedge \psi_2$	$n = \infty$	Increaser	(ψ_1, s, ∞) or (ψ_2, s, ∞)
$\Diamond\psi$	—	Decreaser	(ψ, t, n) for some $t \in sR$
$\Box\psi$	$n < \infty$	(1) Decreaser	$m_t \in \mathbf{N}_\infty$ for each $t \in sR$, such that $\sum m_t = n$
		(2) Increaser	(ψ, t, m_t) for some $t \in sR$
$\Box\psi$	$n = \infty$	(1) Increaser	Either (ψ, t, ∞) for some $t \in sR$ or infinite subset $A \subseteq sR$
		(2) Decreaser	$(\psi, t, 0)$ for some $t \in A$, if A is selected in (1)
X	$\sigma_X = \mu$ or $n < \infty$	(either)	$(\mathrm{BFS}(X), s, n)$
X	$\sigma_X = \nu$ and $n = \infty$	Decreaser	$(\mathrm{BFS}(X), s, \infty)$ or $(\mathrm{BFS}(X), s, 0)$
$\sigma X \psi$	—	(either)	(ψ, s, n)

Table 2 Priorities of vertex (φ, s, n)

φ	Side conditions	Priority ($\varphi \in \mathrm{SF}^+$)	Priority ($\varphi \in \mathrm{SF}^-$)
$\mathrm{BFS}(X)$	$\sigma_X = \mu$ or $n = \infty$	$2k_X + 1$	$2k_X$
$\mathrm{BFS}(X)$	$\sigma_X = \nu$ and $n < \infty$	$2k_X$	$2k_X + 1$
X	$\sigma_X = \nu$, $n = \infty$, and $\exists\psi.\ X \leq \neg\psi \leq \mathrm{BF}(X)$	$2k_X + 2$	$2k_X + 1$
all others	—	0	0

In particular, $\gamma(\varphi, \infty)$ is either $\mathbf{N}_\infty \setminus \{\infty\}$ or $\{\infty\}$, depending on whether $\varphi \in \mathrm{SF}^+$ or $\varphi \in \mathrm{SF}^-$. The intuitive meaning of vertex (φ, s, n) is "the value of φ at s belongs to $\gamma(\varphi, n)$." Player tries to claim this statement, while Opponent tries to rebut it. We call Player "Decreaser" and Opponent "Increaser" when $\varphi \in \mathrm{SF}^+$, and Player "Increaser" and Opponent "Decreaser" when $\varphi \in \mathrm{SF}^-$.

There are three types of end vertices:

- (p, s, n), where p is a propositional symbol. If a play reaches here, Player wins if $L(p) \in \gamma(p, n)$, Opponent wins otherwise.
- $(\Box\varphi, s, n)$, with $sR = \varnothing$. Player wins here.
- $(\Diamond\varphi, s, n)$, with $sR = \varnothing$. Opponent wins here.

The possible moves at other vertices are defined in Table 1.

For each variable X, we assign $k_X \in \mathbf{N}$ so that $k_Y < k_X$ if $\mathrm{BF}(Y) < \mathrm{BF}(X)$. Then, the priority of vertices is defined in Table 2. If a play is infinite, Player wins if $\max\{n \mid$ there are infinitely many i such that n is the priority of the i-th vertex$\}$ is an even number. Otherwise, Opponent wins.

3.2 Strategy

We define a memoryless strategy for Player and Opponent. Assume that the current position is (φ, s, n).

First, we define $\xi = \mathrm{lseq_D}(\varphi, s, n)$ for the strategy of Decreaser and $\xi = \mathrm{lseq_I}(\varphi, s, n)$ for the strategy of Increaser. If ξ is not defined, then the corresponding player has no chance in winning, and takes any legal move. In the following, we assume that ξ is defined.

Case ($\varphi = \neg\psi$). There is no option to choose.

Case ($\varphi = \psi_1 \vee \psi_2$). Decreaser selects (ψ_1, s, n) if $\mathrm{iv}(\psi_1, \xi, s) \leq \mathrm{iv}(\psi_2, \xi, s)$, and (ψ_2, s, n) otherwise.

Case ($\varphi = \psi_1 \wedge \psi_2$ and $n < \infty$). Let $m'_j = \mathrm{iv}(\psi_1, \xi, s)$ for $j = 1, 2$. If $m'_1 + m'_2 < n + 1$, Decreaser takes $m_1 = m'_1$ and $m_2 = n - m_1$; otherwise, he has no chance. Once m_1 and m_2 are selected, if there is $j \in \{1, 2\}$ such that $m'_j \geq m_j + 1$, Increaser selects (ψ_j, s, m_j) for this j; otherwise, he has no chance.

Case ($\varphi = \psi_1 \wedge \psi_2$ and $n = \infty$). If there is a variable that is singular in φ and s, then let X be the $\prec$-least such variable. By Lemma 4 (4), there exists $j \in \{1, 2\}$ such that X is singular in φ_j and s Increaser selects (φ_j, s, ∞) for this j. If there is no singular variable, and there is $j \in \{1, 2\}$ such that $\mathrm{iv}(\psi_j, \xi, s) = \infty$, Increaser selects (φ_j, s, ∞) for this j. Otherwise, he has no chance.

Case ($\varphi = \Diamond\psi$). Let $t \in sR$ be the one that gives the least value of $\mathrm{iv}(\psi, \xi, t)$. Decreaser selects (ψ, t, n).

Case ($\varphi = \Box\psi$ and $n < \infty$). Let $m'_t = \mathrm{iv}(\psi, \xi, t)$ for $t \in sR$. If $\sum m'_t \leq n$, Decreaser takes m_t so that $m_t \geq m'_t$ and $\sum m_t = n$. otherwise, he has no chance. Once m_t is selected, if there is $t \in sR$ such that $m'_t \geq m_t + 1$, Increaser selects that (ψ, t, m_t); otherwise, either of the candidates is selected.

Case ($\varphi = \Box\psi$, and $n = \infty$). If there is a variable that is singular in φ and s, then let X be the $\prec$-least such variable and $l = \mathrm{idx}(X)$. By Lemma 4 (6), either there exists t such that X is singular in ψ and s, or there exists an infinite $A \subseteq sR$ such that $\mathrm{lseq_I}(\psi, t, 0) \upharpoonright (l+1) = \xi \upharpoonright (l+1)$ for all $t \in A$. Decreaser selects (ψ, t, ∞) in the former case, and A in the latter case.

In the case where there is no singular variable, Increaser checks if there is $t \in sR$ such that $\mathrm{iv}(\psi, \xi, t) = \infty$. If there is, he selects (ψ, t, ∞) for this t. If there is not, but $A = \{t \in sR \mid \mathrm{iv}(\psi, \xi, t) \geq 1\}$ is infinite, Increaser selects this A. Otherwise, he has no chance.

If an infinite set A is selected by Increaser, Decreaser checks if there is $t \in A$ such that $\mathrm{iv}(\psi, \xi, t) = 0$. If there is, he selects $(\psi, t, 0)$ for this t. Otherwise, he has no chance.

Case ($\varphi = X$). If $\sigma_X = \mu$ or $n < \infty$, then there is no options to select. Otherwise, i.e., $\sigma_X = \nu$ and $n = \infty$, Decreaser selects $(\mathrm{BFS}(X), s, 0)$ if $\mathrm{iv}(X, \xi, s) = 0$, and selects $(\mathrm{BFS}(X), s, \infty)$ otherwise.

Case ($\varphi = \sigma X \psi$). There is no option to select.

That completes the definition of the strategy.

3.3 Examples

Let $\mathcal{K}_1 = (S_1, R_1, L_1)$ be a Kripke structure defined as follows:

- $S_1 = \{s_1, s_2\}$
- $R_1 = \{(s_1, s_1), (s_1, s_2)\}$
- $L_1(p, s_1) = \infty$, $L_1(p, s_2) = 1$.

Assume that the given formula φ_I is $\mu X(p \vee \Diamond X)$, and let $\varphi_0 = p \vee \Diamond X$. Following the definition, we can compute some intermediate values $\mathrm{iv}(\psi, \xi, s)$ as in Table 3. Thus, we know that $[\![\varphi_\mathrm{I}]\!]^{\mathcal{K}_1}(s_1) = 1$. Let us see how Player and Opponent wins the game at $(\varphi_\mathrm{I}, s_1, 1)$ and $(\varphi_\mathrm{I}, s_1, 0)$, respectively, by obeying the strategy.

In the first game, the first vertex is $(\varphi_\mathrm{I}, s_1, 1)$. There is no options and the next vertex is $(\varphi_0, s_1, 1)$. Since Player obeys the strategy, he calculate $\mathrm{lseq_I}(\varphi_0, s_1, 1)$ and gets $\xi = 1$, because $\mathrm{iv}(\varphi_0, 1, s_1) < 1 + 1$ and $\mathrm{iv}(\varphi_0, 0, s_1) \not< 1 + 1$. Because $\varphi_0 = p \vee \Diamond X$, it is Player's turn. He computes $\mathrm{iv}(p, 1(=\xi), s_1) = \infty$ and $\mathrm{iv}(\Diamond X, 1, s_1) = 1$. Therefore, he selects $(\Diamond X, s_1, 1)$. Again, it is Player's turn. After confirming that $\xi = \mathrm{lseq_I}(\Diamond X, s_1, 0) = 1$, he compares $\mathrm{iv}(X, 1, s_1) = \infty$ and $\mathrm{iv}(X, 1, s_2) = 1$, therefore $(X, s_2, 1)$ is selected. The play continues like this, and finally, the game reaches $(p, s_2, 1)$. Because $L_1(p, s_2) = 1 < 1 + 1$, Player wins in this play.

In the second game, the first vertex is $(\varphi_\mathrm{I}, s_1, 0)$ and the second vertex is $(\varphi_0, s_1, 0)$ as in the first game. Here, opponent computes $\xi = \mathrm{lseq_D}(\varphi_0, s_1, 0)$. Because $X \in \mathrm{FV_D}$, $\mathrm{Seq_D}$ is a singleton and its only element is κ, which, in this game, can be any ordinal number greater than 0, $\xi = \kappa$. However, Opponent does not have a chance to use this ξ, because it is Player's turn. Because Player does not obey the strategy, he can select either of $(p, s_1, 0)$ or $(\Diamond X, s_1, 0)$ as he likes. Probably, he will select the latter, as otherwise he would immediately lose this play. In each turn, as in this turn, Player virtually has no option if he avoids an immediate loss, and play continues for ever as follows: $(\varphi_\mathrm{I}, s_1, 0)$, $(\varphi_0, s_1, 0)$, $(\Diamond X, s_1, 0)$, $(X, s_1, 0)$, $(\varphi_0, s_1, 0)$, $\cdots$. Among these vertices, only $(\varphi_0, s_1, 0)$ has a positive priority 1, which is an odd number. (Here, we assume $k_X = 0$, but any other number also works in the same way.) Therefore, Opponent wins.

As another example, let $\mathcal{K}_2 = (S_1, R_1, L_2)$, $L_2(p, s_1) = 1$ and $L_2(p, s_2) = 1$. Assume $\varphi_\mathrm{I} = \nu X(p \wedge \Box X)$, then $[\![\varphi_\mathrm{I}]\!](s_1) = \infty$ and Opponent wins the game starting $(\varphi_\mathrm{I}, s_1, \infty)$ by obeying the strategy, in a similar manner as in the

Table 3 Intermediate Values on $\mathcal{K}_1$

ψ	ξ	$\mathrm{iv}(\psi, \xi, s_1)$	$\mathrm{iv}(\psi, \xi, s_2)$
p	0	∞	1
X	0	∞	∞
$\Diamond X$	0	∞	∞
φ_0	0	∞	1
X	1	∞	1
$\Diamond X$	1	1	∞
φ_0	1	1	1
φ_0	2	1	1
φ_I	ϵ	1	1

previous example. Let us now make a small modification and assume that $\varphi_{\rm I} = \nu X(p \wedge \neg\Diamond\neg X)$. It is easy to see that $[\![\varphi_{\rm I}]\!]^{\mathcal{K}_2}(s_1) = 1$. If Player obeys the strategy and Opponent avoids an immediate loss, a play goes on as follows: $(\varphi_{\rm I}, s_1, \infty)$, (φ_2, s_1, ∞), $(\neg\Diamond\neg X, s_1, \infty)$, $(\Diamond\neg X, s_1, 0)$, $(\neg X, s_1, 0)$, (X, s_1, ∞), (φ_2, s_1, ∞), $\cdots$, where $\varphi_2 = p \wedge \neg\Diamond\neg X = \mathrm{BFS}(X)$. Vertices that have positive priorities are (φ_2, s_1, ∞) and (X, s_1, ∞). The priorities of the former is 1 and the latter is 2. Therefore, Player wins the play.

3.4 Invariants

Let $(\varphi_i, s_i, n_i)_{i<\zeta}$ be the vertices visited during a play, where ζ is the length of a play, either a positive natural number or ω. In the rest of the paper, we always assume the following:

- $(\varphi_0, s_0, n_0) = (\varphi_{\rm I}, s_{\rm I}, n_{\rm I})$.
- If $[\![\varphi_{\rm I}]\!](s_{\rm I}) < n_{\rm I}+1$, Player obeys the strategy.
- If $[\![\varphi_{\rm I}]\!](s_{\rm I}) \geq n_{\rm I}+1$, Opponent obeys the strategy.

We define $\xi_i = \mathrm{lseq_D}(\varphi_i, s_i, n_i)$ if either Player obeys the strategy and $\varphi_i \in \mathrm{SF}^+$, or Opponent obeys the strategy and $\varphi_i \in \mathrm{SF}^-$. Otherwise, $\xi_i = \mathrm{lseq_I}(\varphi_i, s_i, n_i)$. In both cases, ξ_i is the one that is used to decide the next move by the player who obeys the strategy.

Lemma 5.

(1) If $[\![\varphi_{\rm I}]\!](s_{\rm I}) < n_{\rm I} + 1$, then $\mathrm{iv}(\varphi_i, \xi_i, s_i) \in \gamma(\varphi_i, n_i)$ for all $i < \zeta$.

(2) If $[\![\varphi_{\rm I}]\!](s_{\rm I}) \geq n_{\rm I} + 1$, then $\mathrm{iv}(\varphi_i, \xi_i, s_i) \notin \gamma(\varphi_i, n_i)$ for all $i < \zeta$.

(3) Unless $\varphi_i \in \mathrm{PV}$, $\xi_{i+1} \leq \xi_i$.

(4) If $\varphi_i = X \in \mathrm{PV}$, $\xi_{i+1} \leq \xi_i \restriction (\mathrm{idx}(X) + 1)$.

Proof. Let $\xi' = \xi_i \restriction (\mathrm{idx}(X) + 1)$ if $\varphi_i = X \in \mathrm{PV}$, and $\xi' = \xi_i$ otherwise. To prove from (1) to (4), what we need to show are the following:

- If $[\![\varphi_{\rm I}]\!](s_{\rm I}) < n_{\rm I} + 1$, then $\mathrm{iv}(\varphi_i, \xi, s_i) \in \gamma(\varphi_i, n_i) \implies \mathrm{iv}(\varphi_{i+1}, \xi', s_i) \in \gamma(\varphi_{i+1}, n_{i+1})$.
- If $[\![\varphi_{\rm I}]\!](s_{\rm I}) \geq n_{\rm I} + 1$, then $\mathrm{iv}(\varphi_i, \xi, s_i) \notin \gamma(\varphi_i, n_i) \implies \mathrm{iv}(\varphi_{i+1}, \xi', s_i) \notin \gamma(\varphi_{i+1}, n_{i+1})$.

If $i = 0$, the conclusion follows from Lemma 1. The $(i+1)$ cases can be confirmed by checking each step of the strategy. Here, we only show two cases. The other cases can be shown in a similar manner.

Case $(\varphi_i = \psi_1 \vee \psi_2 \in \mathrm{SF}^+)$. Let $s = s_i = s_{i+1}$ and $n = n_i = n_{i+1}$.

First assume $[\![\varphi_{\rm I}]\!](s_{\rm I}) < n_{\rm I} + 1$ and $\mathrm{iv}(\varphi_i, \xi_i, s) < n + 1$. Without loss of generality, we can assume $\mathrm{iv}(\psi_1, \xi_i, s) \leq \mathrm{iv}(\psi_2, \xi_i, s)$. According to the strategy, $(\varphi_{i+1}, s, n) = (\psi_1, s, n)$. That means $\mathrm{iv}(\varphi_{i+1}, \xi_i, s) = \mathrm{iv}(\varphi_i, \xi_i, s) < n + 1$.

Next assume $[\![\varphi_{\rm I}]\!](s_{\rm I}) \geq n_{\rm I}+1$ $\mathrm{iv}(\varphi_i, \xi_i, s) \geq n+1$, and Player selects ψ_k. We have $\mathrm{iv}(\psi_k, \xi_i, s) \geq \mathrm{iv}(\varphi_i, \xi_i, s) \geq n + 1$.

Case $(\varphi_i = X \in \mathrm{SF}^-$, $\mathrm{BF}(X) = \nu X\psi$, and $n_i = \infty)$. Let $s = s_i = s_{i+1}$ and $\xi' = \xi_i \restriction (\mathrm{idx}(X) + 1)$.

First assume that $[\![\varphi_{\rm I}]\!](s_{\rm I}) < n_{\rm I} + 1$ and $\mathrm{iv}(X, \xi_i, s) = \infty$. By definition, we have $\mathrm{iv}(\psi, \xi', s) = \infty$. In this case $n_{i+1} = \infty$ or $n_{i+1} = 0$, depending on Opponent's choice. But in either case, we have $\mathrm{iv}(\varphi_{i+1}, \xi', s) = \infty \geq n_{i+1} + 1$.

Next assume that $[\![\varphi_{\rm I}]\!](s_{\rm I}) \geq n_{\rm I} + 1$ and $\mathrm{iv}(X, \xi_i, s) < \infty$. Because $\xi_i \in \mathrm{Seq_D}(X)$ and $X \in \mathrm{FV_I}(X)$, we have $\xi_i(l) = \kappa$. Therefore, $\mathrm{iv}(\psi, \xi', s) = \mathrm{iv}(X, \xi_i, s)$. If $\mathrm{iv}(X, \xi_i, s) = 0$, Opponent selects $(\psi, s, 0)$ and we have $\mathrm{iv}(\psi, \xi', s) = 0 < 1$. Otherwise, Opponent selects (ψ, s, ∞) and we have $\mathrm{iv}(\psi, \xi', s) < \infty$. □

3.5 Infinite Plays

In order to analyze finite plays, Lemma 5 is sufficient. From now on, until the end of this section, we assume that plays are infinite, namely $\zeta = \omega$.

It is clear that the following lemma holds.

Lemma 6.

(1) There is a propositional variable X that occurs infinitely many times as φ_i, that is, there are infinitely many $i \in \mathbf{N}$ such that $\varphi_i = X$.

(2) If X and Y are propositional variables that satisfy (1), then there exists propositional variable Z that also satisfies (1), $X \preceq Z$, and $Y \preceq Z$. □

Therefore, for every play, there exists a $\prec$-maximum propositional variable that occur infinitely many times as φ_i. We call it the *principal variable* of the play and denote it by $X_{\rm P}$. The index $\mathrm{idx}(X_{\rm P})$ of the principal variable is denoted by $l_{\rm P}$. Let $\psi_{\rm P} = \mathrm{BFS}(X_{\rm P})$.

Lemma 7. *There exist $I_{\rm P} \in \mathbf{N}$ and $\xi_{\rm P} \in \mathrm{Seq}_{\psi_{\rm P}}$ such that for all $i \geq I_{\rm P}$,*

- $\varphi_i = X \in \mathrm{PV} \implies X \preceq X_{\rm P}$.
- $\xi_i \restriction (l_{\rm P} + 1) = \xi_{\rm P}$.

Proof. Clear from the definition of the principal variable and Lemma 5. □

Lemma 8.
(1) If $\sigma_{X_P} = \mu$, then $X_P \in SF^+$ if and only if $[\![\varphi_I]\!](s_I) \geq n_I + 1$.
(2) If $\sigma_{X_P} = \nu$ and there is no $i \geq I_P$ such that $\varphi_i = X_P$ and $iv(X_P, \xi_i, s_i) = \infty$, then $X_P \in SF^+$ if and only if $[\![\varphi_I]\!](s_I) < n_I + 1$.
(3) If $\sigma_{X_P} = \nu$ and there exists $i \geq I_P$ such that $\varphi_i = X_P$ and $iv(X_P, \xi_i, s_i) = \infty$, then $X_P \in SF^+$ if and only if $[\![\varphi_I]\!](s_I) \geq n_I + 1$.

Proof. We only prove the lemma when $X_P \in SF^+$. The other case, $X_P \in SF^-$, can be shown in a similar manner.
(1) Assume $\sigma_{X_P} = \mu$ and $[\![\varphi_I]\!](s_I) < n_I + 1$. If $\varphi_i = X_P$, then $\xi_i = \mathrm{lseq}_D(\varphi_i, s_i, n_i)$. Therefore, $\xi_i \restriction (l_P + 1) > \xi_{i+1}$. This cannot happen since there are infinitely many such that $\varphi_i = X_P$.
(2) Similar to case (1).
(3) Note that $iv(X_P, \xi_{i_0}, s_{i_0}) = \infty$ implies $iv(X_P, \xi_{i_0}, s_{i_0}) \notin \gamma(\infty)$ because $X_P \in SF^+$. Therefore, $[\![\varphi_I]\!](s_I) \geq n_I + 1$ by Lemma 5. □

Lemma 9. *Assume $\sigma_{X_P} = \nu$, $i_0 \geq I_P$, $\varphi_{i_0} = X_P$, and $iv(X_P, \xi_{i_0}, s_{i_0}) = \infty$. Then, for all $i \geq i_0$, there exists a variable $X \succeq X_P$ that is singular in φ_i and s_i.*

Proof. We prove the lemma by induction on i.

First, consider the case $i = i_0$. Let $\alpha = \xi_{i_0}(l_P)$ and $\psi = BFS(X_P)$. Since $iv(X_P, \xi_{i_0}, s_{i_0}) = \infty$, $\alpha > 0$. If $\alpha = \beta + 1$, then by the definition of iv, $iv(\psi, (\xi_{i_0} \restriction (l_P + 1)) : \beta, s_{i_0}) = \infty$, and hence $\xi_{i_0+1} < \xi_{i_0} \restriction (l_P + 1)$, which contradicts $i_0 \geq I_P$. Therefore, α is a limit ordinal. For $k \in \mathbf{N}$, let β_k be the least ordinal such that $iv(\psi, \xi_{i_0} : \beta_k, s_{i_0}) \geq k$. Again, since $i_0 \geq I_P$, we have $\beta_k < \alpha$. Therefore, by taking $\eta_k = \xi_{i_0} : \beta_k$, we can confirm that X_P is singular in φ_{i_0} and s_{i_0}.

Next, the case $i + 1$. We assume that there is a variable $X \succeq X_P$ that is singular in φ_i and s_i. Depending on the form of φ_i, we can use the corresponding item of Lemma 4 to confirm that X is also singular in φ_{i+1} and s_{i+1}. Note that by Lemma 4 (2), φ_i cannot be in the form of $\neg\psi$. Also, because $i \geq I_P$ and $X \succeq X_P$, we cannot have $\mathrm{lseq}_I(\cdot, \cdot, \infty) \restriction (l+1) < \xi_i \restriction (l+1)$ appearing in the conclusions of Lemma 4. □

Lemma 10. *Assume that $\sigma_{X_P} = \nu$ and there is $I \geq I_P$ such that:*
- *$iv(X_P, \xi_I, s_I) < \infty$.*
- *φ_i is not in the form of $\neg\psi$ for all $i \geq I$.*

Then, there is $J \geq I$ such that $iv(\varphi_i, \xi_i, s_i) = 0$ for all $i \geq J$.

Proof. By checking the strategy, we can easily confirm that $iv(\varphi_i, \xi_i, s_i) \geq iv(\varphi_{i+1}, \xi_{i+1}, s_{i+1})$ for all $i \geq I$. (Note that the negation operator does not appear.) Therefore, there are $J \geq I$ and $c < \infty$ such that for all $i \geq J$, $iv(\varphi_i, \xi_i, s_i) = c$. Without loss of generality, we assume $\varphi_J = X_P$.

Let $T = \{i \geq J \mid \varphi_i = X \in PV, \sigma_X = \nu\}$ and $U = \{(i, \eta) \mid i \geq J, \eta \in \mathrm{Seq}_D(\varphi_i), \eta \restriction l_P = \xi_P \restriction l_P$, and $\eta \leq \xi_i\}$. We prove that $iv(\varphi_i, \eta, s_i) = 0$ for all $(i, \eta) \in U$. This is sufficient for the conclusion of the lemma. We use induction: roughly, the order of the induction for particular η is that the case $i \in T$ is shown first, and then cases $i-1, i-2, \ldots$ are shown until $i-k \in T$; and this cycle is repeated with η increasing. Formally, the induction is performed with the following well-founded relation W on U: $((i, \eta), (i', \eta')) \in U$ if and only if either $i \in T$ and $\eta < \eta'$, or $i = i' + 1$, $i' \notin T$, and $\eta \leq \eta'$.

Case ($\varphi_i = X \in PV$ and $\sigma_X = \nu$). By the definition of $iv(X, \cdot, \cdot)$ and induction hypothesis, we have $iv(X_P, \eta, s_i) = 0$.

Case ($\varphi_i = X \in PV$ and $\sigma_X = \mu$). Because $iv(X, \eta, s_i) = iv(BFS(X), \eta, s_i)$, we have $iv(X, \eta, s_i) = 0$ by induction hypothesis. Note that $BFS(X) = \varphi_{i+1}$.

Case ($\varphi_i = \psi_1 \vee \psi_2$ or $\varphi_i = \Diamond\psi$). For the first half, without loss of generality, we assume $\varphi_{i+1} = \psi_1$. Then, $iv(\varphi_i, \eta, s_i) \leq iv(\psi_1, \eta, s_i) = 0$ by induction hypothesis. The second half is similar.

Case ($\varphi_i = \psi_1 \wedge \psi_2$ or $\varphi_i = \Box\psi$). For the first half, without loss of generality, we assume $\varphi_{i+1} = \psi_1$. Then, $c = iv(\varphi_i, \xi_i, s_i) = iv(\varphi_{i+1}, \xi_i, s_i) + iv(\psi_2, \xi_i, s_i) \geq c + iv(\psi_2, \eta, s_i)$. Therefore, $iv(\psi_2, \eta, s_i) = 0$. On the other hand, $iv(\psi_1, \eta, s_i) = 0$ by induction hypothesis. Thus, $iv(\varphi_i, \eta, s_i) = 0$. The second half is similar.

Case ($\varphi_i = \sigma Y \psi$). In this case, $Y \prec X$ and for some suitable ordinal number α, $iv(\varphi_i, \eta, s_i) = iv(\psi, \eta{:}\alpha, s_i) = iv(\varphi_{i+1}, \eta{:}\alpha, s_{i+1}) = 0$ by induction hypothesis. □

4 Equivalence

Now, we are ready to prove the main theorem.

Theorem 11. $[\![\varphi_\mathrm{I}]\!](s_\mathrm{I}) < n_\mathrm{I} + 1$ *if and only if* $(\varphi_\mathrm{I}, s_\mathrm{I}, n_\mathrm{I})$ *belongs to the winning region of Player.*

Proof. It is sufficient to show the following two claims.

- If $[\![\varphi_\mathrm{I}]\!](s_\mathrm{I}) < n_\mathrm{I}+1$ and Player obeys the strategy shown in the previous section, then Player wins.
- If $[\![\varphi_\mathrm{I}]\!](s_\mathrm{I}) \geq n_\mathrm{I} + 1$ and Opponent obeys the strategy, then Opponent wins.

By Lemma 5, these claims are correct for all finite plays, because $\mathrm{iv}(p, \xi_i, s_i) = L(p)$ for $p \in \mathrm{PS}$, and if $s_i R = \varnothing$, then $\mathrm{iv}(\Diamond\psi, \xi_i, s_i) = \infty$ and $\mathrm{iv}(\Box\psi, \xi_i, s_i) = 0$.

Assume we have an infinite play $(\varphi_i, s_i, n_i)_{i<\omega}$. Let X_P, l_P, I_P and ξ_i be defined as in the previous section. We only show when $X_\mathrm{P} \in \mathrm{SF}^+$; the other case, $X_\mathrm{P} \in \mathrm{SF}^-$, can be shown in a similar argument. According to Lemma 8, we consider three cases:

(1) $\sigma_{X_\mathrm{P}} = \mu$.

(2) $\sigma_{X_\mathrm{P}} = \nu$ and there is no $i \geq I_\mathrm{P}$ such that $\varphi_i = X_\mathrm{P}$ and $\mathrm{iv}(X_\mathrm{P}, \xi_i, s_i) = \infty$.

(3) $\sigma_{X_\mathrm{P}} = \nu$ and there exists $i \geq I_\mathrm{P}$ such that $\varphi_i = X_\mathrm{P}$ and $\mathrm{iv}(X_\mathrm{P}, \xi_i, s_i) = \infty$.

In case (1), we have $[\![\varphi_\mathrm{I}]\!](s_\mathrm{I}) \geq n_\mathrm{I} + 1$ by Lemma 8. Priority of the play is $2k_{X_\mathrm{P}}+1$, therefore Opponent wins.

In case (2), we have $[\![\varphi_\mathrm{I}]\!](s_\mathrm{I}) < n_\mathrm{I}+1$. If the negation operator appear infinitely many times, then the priority of the play is $2k_{X_\mathrm{P}} + 2$ and Player wins. Otherwise, by Lemma 10, $\varphi_i = X_\mathrm{P}$ and $\mathrm{iv}(\varphi_i, \xi_i, s_i) = 0$ for some i. At this point, according to the strategy, Player selects so that $n_{i+1} = 0$. Therefore, the priority of the play is $2k_{X_\mathrm{P}}$, and Player wins.

In case (3), we have $[\![\varphi_\mathrm{I}]\!](s_\mathrm{I}) \geq n_\mathrm{I} + 1$ By Lemmas 9 and 4 (2), the negation operator appears only finitely many times in the play and $\mathrm{iv}(\varphi_i, \xi i, s_i) = \infty$ for all $i \leq I_\mathrm{P}$ such that $\varphi_i = X_\mathrm{P}$. Therefore, the priority of the play is $2k_{X_\mathrm{P}} + 1$, i.e., Opponent wins. □

5 Conclusions

In this paper, we define a parity game between Player and Opponent that characterizes the $\mathbf{N}_\infty$ semantics of the modal μ-calculus: we prove that the value of a formula at a state in a Kripke structure is smaller than an element of $\mathbf{N}_\infty$ if and only if Player has a winning strategy at the corresponding vertex of the game. As future work, we plan to use the game to build a decision procedure for satisfiability of the modal μ-calculus under the semantics.

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