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# Decidability and Undecidability Results of Modal $\mu$-calculi with $\mathbf{N}_\infty$ Semantics

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In our previous study, we defined the semantics of modal $\mu$-calculus on min-plus algebra $\mathbf{N}_\infty$ and developed a model-checking algorithm. $\mathbf{N}_\infty$ is the set of all natural numbers and infinity ($\infty$), and has two operations min and plus. In the semantics, disjunctions are interpreted by min and conjunctions by plus. This semantics allows interesting properties of a Kripke structure, such as the shortest path to some state or the number of states that satisfy a specified condition, to be expressed using simple formulae. In this study, we investigate the satisfiability problem in $\mathbf{N}_\infty$ semantics and prove decidability and undecidability results. First, the problem is decidable if the logic does not contain the implication operator. We prove this result by defining a translation $\mathrm{tr}(\varphi)$ of formula $\varphi$ such that the satisfiability of $\varphi$ in $\mathbf{N}_\infty$ semantics is equivalent to that of $\mathrm{tr}(\varphi)$ in ordinary semantics. Second, the satisfiability problem becomes undecidable if the logic contains the implication operator.

## 1. Introduction

Modal $\mu$-calculus, which uses fixed-point operators, can express various properties of Kripke structures, such as reachability and the existence of infinite paths, both accurately and simply.[6)]

To enhance the expressiveness of this modal logic, attempts have been made to define semantics that interpret formulae on algebra using richer structures than those used in ordinary semantics (i.e., interpreting formulae as true or false).

For example, to formalize multiplexed model-checking, Nishizawa et al. investigated the simulation relation and the relationship between state and path formulae on extended Kripke structures that assign elements of complete Heyting algebra as truth values to propositions and transition relations rather than using ordinary Boolean algebra $\{0, 1\}$.[5)]

We have proposed semantics of modal $\mu$-calculus that interpret disjunctions by min and conjunctions by plus.[4)] Using plus, it is possible to compute or count quantitative measures. To apply this to algebra, we adopted *min-plus algebra,* algebraic structures with two binary operators, min and plus. Their algebraic properties have been extensively studied and they were applied to solve problems in formal language theory, such as finite power property problem.[12)] They are also widely used to analyze discrete event systems, optimization, etc.[2)] An algebraic structure with the following properties is called a dioid:

- min is associative and commutative,
- plus is associative and distributes over min,
- $\infty$ is zero element with respect to min,
- zero element $\infty$ is absorptive with respect to plus,
- plus has unit element 0, and
- min is idempotent.

In addition, if

- plus is commutative,

then it is called a commutative dioid. The algebra $\mathbf{N}_\infty$ that consists of all natural numbers $\mathbf{N}$, including 0 and infinity $\infty$ is a commutative dioid. We interpret modal logic formulae on this algebra $\mathbf{N}_\infty$. A dioid is also known as an idempotent semiring; it satisfies the requirements for a ring except for the existence of inverse elements with respect to min. The algebra that consists of all real numbers (or all integers) and infinity $\infty$ is an idempotent commutative semifield; inverse elements with respect to plus exist except for $\infty$.

We interpret disjunctions by min and conjunctions by plus, so the typical element that represents truth in $\mathbf{N}_\infty$ is 0, and $\infty$ represents falsity. Finite elements other than 0 also represent truth, i.e., there are various levels of truth.

This semantics might remind the reader of the proof-number search in game programming.[1)] The proof number of a node in a game tree means the number of nodes that should be traversed for showing that it is a winning node. The proof number of an AND node is the sum of the proof numbers of its children, while that of an OR node is the minimum among them.

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In our previous study,[4] we formalized two variants of logics, denoted by $\mathcal{L}$ and $\mathcal{L}_{\rightarrow}$ in this paper. The former includes negation, conjunction, disjunction, diamond, square, $\mu$, and $\nu$ operators. The latter is an extension of the former, obtained by adding the implication operator. Unlike ordinary semantics, distributive and De Morgan laws are not satisfied in $\mathbf{N}_\infty$ semantics. We designed model-checking algorithms for the logics according to the semantics, and implemented them efficiently.

On the basis of this semantics, various properties of a Kripke structure can be expressed using formulae. For example, formula $\mu X(p \vee \Diamond(\mathbf{1} \wedge X))$ expresses the length of the shortest path to a state at which $p$ is true, where $\mathbf{1}$ is an atomic formula that is always interpreted as $1 \in \mathbf{N}_\infty$. As another example, formula $[o](\neg q \vee \mathbf{1})$ expresses the number of states at which $q$ is true in the entire Kripke structure, where $o$ is a special modality termed as the universal modality. We had outlined the possibility of application of the expressive power in two directions: shape analysis[10] in the style of 11), and data flow analysis for compiler optimization in the style of 8).

In this work, we investigate the satisfiability problem with respect to $\mathbf{N}_\infty$ semantics. The satisfiability problem in ordinary semantics, that is, the problem of deciding, for given closed formula $\varphi$, whether there is a Kripke structure $\mathcal{K}$ and its state $s$ such that $\mathcal{K}, s \models \varphi$, is decidable and the complexity is EXPTIME-complete.[3] In $\mathbf{N}_\infty$ semantics, the truth value of a formula is represented as an element of $\mathbf{N}_\infty$. Let us denote the truth value of formula $\varphi$ at state $s$ of Kripke structure $\mathcal{K}$ by $[\![\varphi]\!]^{\mathcal{K}}(s)$. Then, the satisfiability problem in $\mathbf{N}_\infty$-semantics version is obtained by replacing $\mathcal{K}, s \models \varphi$ in the above-mentioned definition with $[\![\varphi]\!]^{\mathcal{K}}(s) = 0$. A different version would be obtained using $[\![\varphi]\!]^{\mathcal{K}}(s) < \infty$, but we found that both versions can be treated in a similar way.

There are two main results. One is that the satisfiability problem of $\mathcal{L}$ is decidable. We show this by reducing the problem to the satisfiability problem of the ordinary modal $\mu$-calculus. To achieve this, for formula $\varphi$ of $\mathcal{L}$, we define its translation, namely, formulae $\mathrm{tr}(\varphi, 0)$ and $\mathrm{tr}(\varphi, \infty)$ of the ordinary $\mu$-calculus such that $[\![\mathrm{tr}(\varphi, 0)]\!]^{\mathcal{K}}(t) = 0$ is realized (i.e., $\varphi$ is satisfiable) if and only if $\mathrm{tr}(\varphi, 0)$ is satisfiable and $[\![\mathrm{tr}(\varphi, \infty)]\!]^{\mathcal{K}}(t) = \infty$ is realized if and only if $\mathrm{tr}(\varphi, \infty)$ is satisfiable. The difficulty lies in the case of $\mathrm{tr}(\nu X\varphi, \infty)$. In ordinary semantics, refuting $\nu X\varphi$ (recall that $\infty$ means falsehood) amounts to finding a witness that $\varphi$ is false with finitely many repetitions. This no longer holds for $\mathbf{N}_\infty$ semantics, because an infinite sum of finite values (truth) can be infinite (falsehood). Nevertheless, we can prove that the value of $\nu X\varphi$ is infinite if and only if the following conditions are satisfied: (1) the claim that the value *may* be infinite cannot be refuted through *infinitely* many repetitions and (2) a witness that shows the value *must* be *positive* can be obtained after *finitely* many repetitions. Using this fact, we successfully define the translation and prove that it has the desired properties, although we need to solve problems such as handling of negations and infinite branchings in a Kripke structure.

The other main result is that the satisfiability problem of $\mathcal{L}_{\rightarrow}$ is undecidable. We prove this by reducing Post's correspondence problem[9] of alphabet $\{0, 1\}$. The main reason for this undecidability is that one can compare truth values of two formulae. In fact, we show that the following extension of the satisfiability problem for $\mathcal{L}$ is undecidable: for two given closed formulae $\varphi$ and $\psi$ of $\mathcal{L}$, decide whether there is a Kripke structure $\mathcal{K}$ and its state $s$ such that $[\![\varphi]\!]^{\mathcal{K}}(s) = [\![\psi]\!]^{\mathcal{K}}(s)$. The undecidability result for $\mathcal{L}_{\rightarrow}$ directly follows from this fact.

The remainder of this paper is organized as follows. In Section 2, the variants of the modal $\mu$-calculus considered in this paper are introduced and $\mathbf{N}_\infty$-semantics is defined. In Section 3, the undecidability results are displayed. In Section 4, we define the translation and prove its correctness to show the decidability results. Section 5 concludes the paper.

## 2. Preliminaries

### 2.1 Syntax and semantics

Let PS be the set of propositional symbols and PV be the set of propositional variables. The formulae of language $\mathcal{L}$ is defined as follows:

$$\varphi ::= p \mid \mathbf{1} \mid X \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu X\varphi \mid \nu X\varphi$$

where $p \in \mathrm{PS}$ and $X \in \mathrm{PV}$. All occurrences of $X$ in $\mu X\varphi$ and $\nu X\varphi$ must be positive in $\varphi$. That is, the number of negations of which the occurrence is in the scope must be even.

$\mathcal{K} = (T, R, L)$ is a *Kripke structure* for $\mathcal{L}$ if $T$ is a set, $R \subseteq T \times T$, and $L : \mathrm{PS} \times T \rightarrow \mathbf{N}_\infty$. The set of Kripke structure for $\mathcal{L}$ is denoted by $\mathrm{KS}_{\mathcal{L}}$. $T$, $R$, and

$$\llbracket p \rrbracket^\rho(t) = L(p,t) \qquad \llbracket \mathbf{1} \rrbracket^\rho(t) = 1$$
$$\llbracket X \rrbracket^\rho(t) = \rho(X,t)$$
$$\llbracket \neg\psi \rrbracket^\rho(t) = \begin{cases} 0 & \text{if } \llbracket \psi \rrbracket^\rho(t) = \infty \\ \infty & \text{if } \llbracket \psi \rrbracket^\rho(t) < \infty \end{cases}$$
$$\llbracket \psi_1 \vee \psi_2 \rrbracket^\rho(t) = \min(\llbracket \psi_1 \rrbracket^\rho(t), \llbracket \psi_2 \rrbracket^\rho(t))$$
$$\llbracket \psi_1 \wedge \psi_2 \rrbracket^\rho(t) = \llbracket \psi_1 \rrbracket^\rho(t) + \llbracket \psi_2 \rrbracket^\rho(t)$$
$$\llbracket \Diamond\psi \rrbracket^\rho(t) = \min(\llbracket \psi \rrbracket^\rho(t') \mid (t,t') \in R)$$
$$\llbracket \Box\psi \rrbracket^\rho(t) = \textstyle\sum(\llbracket \psi \rrbracket^\rho(t') \mid (t,t') \in R)$$
$$\llbracket \mu X\psi \rrbracket^\rho(t) = \inf\{F_\alpha(t) \mid \alpha \in \mathrm{On}\}, \text{ where } F_\alpha(t') = \inf\{\llbracket \psi \rrbracket^{\rho[X \mapsto F_\beta]}(t') \mid \beta < \alpha\}$$
$$\llbracket \nu X\psi \rrbracket^\rho(t) = \sup\{F_\alpha(t) \mid \alpha \in \mathrm{On}\}, \text{ where } F_\alpha(t') = \sup\{\llbracket \psi \rrbracket^{\rho[X \mapsto F_\beta]}(t') \mid \beta < \alpha\}$$

**Fig. 1** the value of formulae

$L$ are written as $|\mathcal{K}|$, $\mathcal{K}.R$, and $\mathcal{K}.L$, respectively. A function $\rho : \mathrm{PV} \times T \to \mathbf{N}_\infty$ is called a *valuation*. Then, for formula $\varphi$ of $\mathcal{L}$ and $t \in T$, the value $\llbracket \varphi \rrbracket^{\mathcal{K},\rho}(t) \in \mathbf{N}_\infty$ of $\varphi$ at $t$ is given in Figure 1. $\mathcal{K}$ and/or $\rho$ are omitted if they are clear from the context. In the figure, On is the class of ordinal numbers. For function $f$, $f[a \mapsto b]$ is the function $g$ whose domain is $\mathrm{dom}(f) \cup \{a\}$, and whose values are defined by $g(a) = b$ and $g(x) = f(x)$ for any $x \in \mathrm{dom}(f) \setminus \{a\}$.

Note that the distributive law $\llbracket \varphi \vee (\psi_1 \wedge \psi_2) \rrbracket(t) = \llbracket (\varphi \vee \psi_1) \wedge (\varphi \vee \psi_2) \rrbracket(t)$ does not hold. Also, $\llbracket \neg\Box\varphi \rrbracket(t) = \llbracket \Diamond\neg\varphi \rrbracket(t)$ does not necessarily hold if a state has infinite successors.

Intuitively, value $n \in \mathbf{N}_\infty$ means true if $n < \infty$ and $\infty$ means false. The value $0 \in \mathbf{N}_\infty$ represents the absolute truth.

Formula **false** and **true** are abbreviations for $p \vee \neg p$ (for some fixed $p \in \mathrm{PS}$) and $\neg$**false**, respectively. Clearly, we have $\llbracket \mathbf{true} \rrbracket(t) = 0$ and $\llbracket \mathbf{false} \rrbracket(t) = \infty$.

Let $\mathcal{L}_\to$ be the language based on $\mathcal{L}$ and expanded by allowing constructing formula $\varphi \to \psi$ from formulae $\varphi$ and $\psi$. Its meaning $\llbracket \varphi \to \psi \rrbracket(t)$ is given as the least $n \in \mathbf{N}_\infty$ such that $\llbracket \varphi \rrbracket(t) + n \geq \llbracket \psi \rrbracket(t)$. Thus, we have $\llbracket \neg\varphi \rrbracket(t) = \llbracket \varphi \to \mathbf{false} \rrbracket(t)$ in $\mathcal{L}_\to$.

A formula $\varphi$ of $\mathcal{L}$ is *satisfiable* if there exists a Kripke structure $\mathcal{K}$ for $\mathcal{L}$, its state $t$, and valuation $\rho$ such that $\llbracket \varphi \rrbracket^{\mathcal{K},\rho}(t) = 0$.

We introduce four "abstract" values of $\mathbf{N}_\infty$: Zer, Fin, Pos, and Inf. Their meanings are given by $\gamma(\mathrm{Zer}) = \{0\}$, $\gamma(\mathrm{Inf}) = \{\infty\}$, $\gamma(\mathrm{Pos}) = \mathbf{N}_\infty \setminus \{0\}$, and $\gamma(\mathrm{Fin}) = \mathbf{N}_\infty \setminus \{\infty\}$. We denote the set of four abstract values by AV, the set $\{\mathrm{Zer}, \mathrm{Fin}\}$ of two less values by AVL, and the set $\{\mathrm{Pos}, \mathrm{Inf}\}$ of two greater values by AVG.

Let $a \in \mathrm{AV}$. a closed formula $\varphi$ of $\mathcal{L}$ or $\mathcal{L}_\to$ is *a-satisfiable* if there is $\mathcal{K} \in \mathrm{KS}(\mathcal{L})$ and $t \in |\mathcal{K}|$ such that $\llbracket \varphi \rrbracket^{\mathcal{K}}(t) \in \gamma(a)$. Formula $\varphi$ is *satisfiable* if it is Zer-satisfiable.

Next, we introduce language $\mathcal{L}'$ as follows. The set $\mathrm{PS}'$ of propositional symbols of $\mathcal{L}'$ is $\{p_0 \mid p \in \mathrm{PS}\} \cup \{p_\infty \mid p \in \mathrm{PS}\}$. The set $\mathrm{Mod}'$ of modality symbols of $\mathcal{L}'$ is $\{1, \infty\}$. Its formulae are defined by:

$$\varphi ::= p' \mid X \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle m \rangle\varphi \mid [m]\varphi \mid \mu X\varphi \mid \nu X\varphi$$

where $p' \in \mathrm{PS}'$, $m \in \mathrm{Mod}'$, and $X$ ranges over the propositional variables. The semantics of $\mathcal{L}'$ is given by Kripke structures $\mathcal{K}' = (S, R', L')$ where $S$ is a set, $R' : \mathrm{Mod}' \to \mathcal{P}(S \times S)$, and $L' : \mathrm{PS}' \to \mathcal{P}(S)$. The satisfaction relation $\mathcal{K}', s \models \varphi'$ for $s \in S$ and formula $\varphi'$ of $\mathcal{L}'$ is defined in an ordinary manner.[6),14)] In particular, we have all classical relations such as $\llbracket \neg(\varphi' \vee \psi') \rrbracket^{\mathcal{K}'} = \llbracket \neg\varphi' \wedge \neg\psi' \rrbracket^{\mathcal{K}'}$, $\llbracket \neg\langle m \rangle\varphi' \rrbracket^{\mathcal{K}'} = \llbracket [m]\neg\varphi' \rrbracket^{\mathcal{K}'}$, or $\llbracket \neg\mu X\varphi' \rrbracket^{\mathcal{K}'} = \llbracket \nu X \neg\varphi'[\neg X/X] \rrbracket^{\mathcal{K}'}$, where we write $\llbracket \varphi' \rrbracket^{\mathcal{K}'}$ for $\{s \in S \mid \mathcal{K}', s \models \varphi'\}$. The set of Kripke structure for $\mathcal{L}'$ is denoted by $\mathrm{KS}_{\mathcal{L}'}$.

A formula of $\mathcal{L}'$ is in *PNF* if the negation symbol only occurs directly in front of propositional symbols. From the classical relations mentioned above, it is obvious that for each formula $\varphi$ of $\mathcal{L}'$, there is a formula $\psi$ in PNF of $\mathcal{L}'$ that is equivalent to $\varphi$, i.e., $\llbracket \varphi \rrbracket^{\mathcal{K}'} = \llbracket \psi \rrbracket^{\mathcal{K}'}$ for any Kripke structure $\mathcal{K}'$ for $\mathcal{L}'$.

We introduce a few notations. Let $\varphi$ be a formula of $\mathcal{L}$ or $\mathcal{L}'$.

The set of occurrences of subformulae of $\varphi$ is denoted by $\mathrm{SF}(\varphi)$. For example, the two occurrences of $X$ in formula $\varphi = X \wedge (p \vee X)$ are treated as two different members of $\mathrm{SF}(\varphi)$.

The symbol $\lambda$ is used to stand for either fixed-point operator, $\mu$ or $\nu$. If the binding formula of propositional variable $X$ in a given formula $\varphi$ is $\lambda X\psi$, we denote $\lambda X\psi$ by $\mathrm{BF}(X)$, $\psi$ by $\mathrm{BFS}(X)$, and $\lambda$ by $\lambda_X$. Variable $X$ is called a *$\mu$-variable* (resp. $\nu$-variable) if $\lambda_X = \mu$ (resp. $\lambda_X = \nu$). The set of $\mu$-variables (resp. $\nu$-variables) is denoted by $\mathrm{PV}_\mu$ (resp. $\mathrm{PV}_\nu$). When $\psi_1$ is a subformula of

$\psi_2$, we write $\psi_1 \le \psi_2$. For $X, Y \in \mathrm{PV}$, we write $X \preceq Y$ if $\mathrm{BF}(X) \le \mathrm{BF}(Y)$, and $X \prec Y$ if $X \preceq Y$ and $X \ne Y$.

### 2.2 The game semantics of the ordinary $\mu$-calculus

Let $\mathcal{K}' = (S, R', L')$ be a Kripke structure for $\mathcal{L}'$ and $\varphi_0$ is a closed formula of $\mathcal{L}'$ in PNF. We introduce a game played by Player and Opponent, which is essentially the same as the one defined in 13).

The arena of the game is defined by $A = \{(\varphi, s) \mid \varphi \in \mathrm{SF}(\varphi_0),\ s \in S\}$. Possible moves at position $(\varphi, s) \in A$ is defined in Table 1. In the table, $p$ is a propositional symbol and $m$ is a modality symbol. For each propositional variable $X$ in $\varphi_0$, a natural number $\Omega(X)$ is assigned so that

- $\Omega(X)$ is even if and only if $\lambda_X = \nu$.
- $X_1 \preceq X_2 \implies \Omega(X_1) \le \Omega(X_2)$.

Player wins a play if Opponent cannot move or the play is infinite and $\Omega(X)$ is even, where $X$ is the $\prec$-largest propositional variable that appears infinitely often in the play (note that such a variable always exist).

The following theorem can be proved in a standard way.

**Theorem 1** $\mathcal{K}, s \models \varphi_0$ if and only if $(\varphi_0, s)$ belongs to the winning region of Player.

## 3. Undecidability

In this section, we prove that the problem whether two formulae of $\mathcal{L}$ can have a same value is undecidable. From this fact, it directly follows that the satisfiability problem of logic $\mathcal{L}_{\rightarrow}$ is undecidable.

Let us give names to relating problems. FORMEQ is the following problem: for given formulae $\varphi$ and $\psi$ of $\mathcal{L}$, decide whether there is a Kripke structure $\mathcal{K} = (S, R, L)$ and $s \in S$ such that $[\![\varphi]\!]^{\mathcal{K}}(s) = [\![\psi]\!]^{\mathcal{K}}(s)$. FORMLEQ is the problem obtained from FORMEQ by replacing $[\![\varphi]\!]^{\mathcal{K}}(s) = [\![\psi]\!]^{\mathcal{K}}(s)$ with $[\![\varphi]\!]^{\mathcal{K}}(s) \le [\![\psi]\!]^{\mathcal{K}}(s)$.

We will show that both FORMEQ and FORMLEQ are undecidable by reducing Post's correspondence problem PCP[9)] of alphabet $\{0, 1\}$ to these problems. As an intermediate problem, we introduce EQFIN, which is to decide, for given formulae $\varphi_1, \ldots, \varphi_k$ and $\psi_1, \ldots, \psi_k$ of $\mathcal{L}$, whether there is a Kripke structure $\mathcal{K} = (S, R, L)$ and $s \in S$ such that $[\![\varphi_i]\!]^{\mathcal{K}}(s) = [\![\psi_i]\!]^{\mathcal{K}}(s) < \infty$ for $i = 1, \ldots, k$.

Let $m$ be a natural number and $\varphi$ be a formula in $\mathcal{L}$. We define formula $\boldsymbol{m}$ by $\mathbf{0} = \mathbf{true}$ and $\boldsymbol{m} = \boldsymbol{n} \wedge \mathbf{1}$, where $n = m - 1$. Formula $m * \varphi$ is defined by $0 * \varphi = \mathbf{true}$ and $m * \varphi = ((m-1) * \varphi) \wedge \varphi$. We have $[\![\boldsymbol{m}]\!](t) = m$ and $[\![m * \varphi]\!](t) = m \cdot ([\![\varphi]\!](t))$.

**Lemma 2** EQFIN can be reduced to FORMEQ and FORMLEQ.

**Proof** Assume that formulae $\varphi_i$ and $\psi_i$ are given for $i = 1, \ldots, k$.

Let $\varphi_i' = (\mathbf{2} \wedge \varphi_i) \vee (\neg \varphi_i)$ and $\psi_i' = (\mathbf{2} \wedge \psi_i) \vee (\mathbf{1} \wedge \neg \psi_i)$. Then, we have $[\![\varphi_i']\!](t) = [\![\psi_i']\!](t)$ if and only if $[\![\varphi_i]\!](t) = [\![\psi_i]\!](t) < \infty$.

Let $\varphi'$ be $(\varphi_1' \wedge \psi_1') \wedge \cdots \wedge (\varphi_k' \wedge \psi_k')$ and $\psi'$ be $2 * (\varphi_1' \vee \psi_1') \wedge \cdots \wedge 2 * (\varphi_k' \vee \psi_k')$. Clearly $[\![\varphi']\!](t) \le [\![\psi']\!](t)$ if and only if $[\![\varphi']\!](t) = [\![\psi']\!](t)$ if and only if for all $i = 1, \ldots, k$, $[\![\varphi_i']\!](t) = [\![\psi_i']\!](t)$, which is equivalent to $[\![\varphi_i]\!](t) = [\![\psi_i]\!](t) < \infty$. ∎

Our remaining task is to reduce PCP to EQFIN. Assume that finite number of pairs $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ of words from alphabet $\{0, 1\}$ are given. We need to decide whether there exists a non-empty sequence $i_1, \ldots, i_m$ of indices such that $\alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$. Without loss of generality, we can assume that $(\alpha_i, \beta_i) \ne (\alpha_j, \beta_j)$ if $i \ne j$.

We introduce a few definitions and notations. A formula $\varphi$ is a *condition* if for any Kripke structure $\mathcal{K} = (S, R, L)$ and $s \in S$, $[\![\varphi]\!]^{\mathcal{K}}(s)$ is either 0 or $\infty$. A sequence of states $(s_i)_i$ is a *path* if $(s_i, s_{i+1}) \in R$ for all $i$ such that $s_i$ and $s_{i+1}$ are defined.

For a word $\alpha$, its reversed word is denoted by $\overline{\alpha}$. We define $c(\alpha)$ by the value

**Table 1** Possible moves at $(\varphi, s)$

| $\varphi$ | Turn | Possible moves |
|---|---|---|
| $p$ | Opponent (if $s \in L'(p)$)<br>Player (if $s \notin L'(p)$) | none |
| $\neg p$ | Opponent (if $s \notin L'(p)$)<br>Player (if $s \in L'(p)$) | none |
| $X$ | Player | $(\mathrm{BFS}(X), s)$ |
| $\psi_1 \vee \psi_2$ | Player | $(\psi_1, s)$, $(\psi_2, s)$ |
| $\psi_1 \wedge \psi_2$ | Opponent | $(\psi_1, s)$, $(\psi_2, s)$ |
| $\langle m \rangle \psi$ | Player | $(\psi, s')$ $((s, s') \in R'(m))$ |
| $[m]\psi$ | Opponent | $(\psi, s')$ $((s, s') \in R'(m))$ |
| $\lambda X \psi$ | Player | $(\psi, s)$ |

$$\mathrm{Geq}(p, 0) = \mathbf{true} \qquad \mathrm{Geq}(p, x+1) = \Diamond(\neg\neg p \wedge \mathrm{Geq}(p, x))$$
$$\mathrm{Eq}(p, x) = \mathrm{Geq}(p, x) \wedge \neg \mathrm{Geq}(p, x+1)$$
$$\mathrm{NumTiles} = \nu X(\neg p_{\mathrm{T}} \vee (\neg\neg p_{\mathrm{T}} \wedge \mathbf{1} \wedge \Box X))$$
$$\mathrm{LenTiles} = \nu X(\neg p_{\mathrm{T}} \vee (\neg\neg p_{\mathrm{T}} \wedge \mathbf{1} \wedge \Diamond X))$$
$$\mathrm{Tile}(\alpha, \beta) = \neg\neg p_{\mathrm{T}} \wedge \neg p_{\mathrm{E}} \wedge \mathrm{Eq}(p^c_\alpha, c(\alpha)) \wedge \mathrm{Eq}(p^d_\alpha, d(\alpha)) \wedge \mathrm{Eq}(p^c_\beta, c(\beta)) \wedge \mathrm{Eq}(p^d_\beta, d(\beta))$$
$$\mathrm{StrA} = \mu X((\mathbf{1} \wedge \neg\neg p_{\mathrm{E}}) \vee \textstyle\bigvee_{i=1}^{n}(\mathrm{Tile}(\alpha_i, \beta_i) \wedge c(\alpha_i) \wedge (d(\alpha_i) * \Diamond X)))$$
$$\mathrm{StrB} = \mu X((\mathbf{1} \wedge \neg\neg p_{\mathrm{E}}) \vee \textstyle\bigvee_{i=1}^{n}(\mathrm{Tile}(\alpha_i, \beta_i) \wedge c(\beta_i) \wedge (d(\beta_i) * \Diamond X)))$$

**Fig. 2** Formulae used in EQFIN

of $\overline{\alpha}$ considered as a binary number, and $d(\alpha)$ by $2^{|\alpha|}$, where $|\alpha|$ is the length of $\alpha$. For example, $c(10110) = 01101_2 = 13$ and $d(10110) = 2^5 = 32$. Note that $\alpha$ is uniquely determined from $c(\alpha)$ and $d(\alpha)$.

We introduce several formulae to describe properties of sequences of the pairs. Their definitions are given in Figure 2, where $p_{\mathrm{T}}$, $p_{\mathrm{E}}$, $p^c_\alpha$, $p^d_\alpha$, $p^c_\beta$, and $p^d_\beta$ are different fixed propositional symbols, $p \in \mathrm{PS}$, and $x \in \mathbf{N}$.

Their intended meanings are as follows: $\mathrm{Geq}(p, x)$ has value 0 if there is a path of length $x$ starting from an adjacent state to the current state. $\mathrm{Eq}(p, x)$ has value 0 if $x$ is the maximum length of such paths. They are conditions.

In a Kripke structure, some nodes represents a pair $(\alpha_i, \beta_i)$. Propositional symbol $p_{\mathrm{T}}$ is used to mark states, They form a sequence, and $p_{\mathrm{E}}$ is used to mark the end of the sequence. We need to express that these states actually form a sequence, i.e., they do not branch or form a cycle. For this purpose, we use Formulae NumTiles and LenTiles. Consider an unwound tree of the states that hereditarily satisfy $p_{\mathrm{T}}$ beginning at state $s$. Then, $[\![\mathrm{NumTiles}]\!](s)$ equals to the number of the nodes of the tree and $[\![\mathrm{LenTiles}]\!](s)$ is the depth of the tree. Therefore, the states with $p_{\mathrm{T}}$ form a finite list if and only if $[\![\mathrm{NumTiles}]\!](s) = [\![\mathrm{LenTiles}]\!](s) < \infty$.

When $[\![\mathrm{Tile}(\alpha, \beta)]\!](s) = 0$ for a state $s$, we consider that $s$ represents a pair $(\alpha, \beta)$. Note that for any state $s$, there is at most one such pair $(\alpha, \beta)$.

$[\![\mathrm{StrA}]\!](s) = 1$ if $[\![p_{\mathrm{E}}]\!] < \infty$. On the other hand, if $[\![p_{\mathrm{E}}]\!](s) = \infty$ and $[\![\mathrm{Tile}(\alpha_i, \beta_i)]\!](s) = 0$, $[\![\mathrm{StrA}]\!](s) = c(\alpha_i) + d(\alpha_i) \cdot \min\{[\![\mathrm{StrA}]\!](s') \mid (s, s') \in R\})$. Therefore, the $|\alpha_i|$ least significant bits of the binary expression of

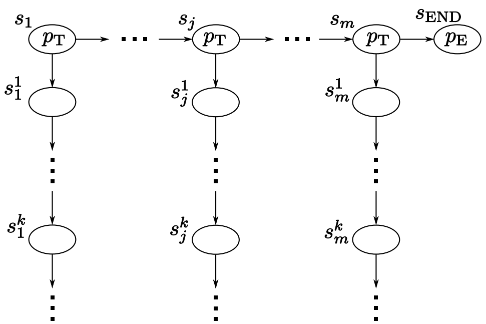


**Fig. 3** Kripke structure for EQFIN

number $[\![\mathrm{StrA}]\!](s)$ is $\overline{\alpha_i}$. Suppose that states $s_1, \ldots, s_m, s_{\mathrm{END}}$ forms a list, $1 \leq i_1, \ldots, i_m \leq n$, $[\![\mathrm{Tile}(\alpha_{i_j}, \beta_{i_j})]\!](s_j) = 0$ for $j = 1, \ldots, m$, $[\![p_{\mathrm{E}}]\!](s_{\mathrm{END}}) = 0$, $[\![p_{\mathrm{T}}]\!](s) = \infty$ if $s \notin \{s_1, \ldots, s_m\}$, and $[\![p_{\mathrm{E}}]\!](s) = \infty$ if $s \neq s_{\mathrm{END}}$. Then, the binary expression of $[\![\mathrm{StrA}]\!](s_1)$ is $1\overline{\alpha_{i_m} \cdots \alpha_{i_1}}$ and that of $[\![\mathrm{StrB}]\!](s_1)$ is $1\overline{\beta_{i_m} \cdots \beta_{i_1}}$.

**Lemma 3** PCP can be reduced to EQFIN.

**Proof** We only give a proof sketch here, based on the intended meanings of the formulae mentioned above.

As an instance of EQFIN, we consider the following pairs of formulae: $(\varphi_1, \psi_1) = (\mathrm{NumTiles}, \mathrm{LenTiles})$, $(\varphi_2, \psi_2) = (\mathrm{StrA}, \mathrm{StrB})$, and $(\varphi_3, \psi_3) = (\neg p_{\mathrm{E}}, \mathbf{true})$.

Suppose that there is a Kripke structure $\mathcal{K} = (S, R, L)$ and $s \in S$ such that $[\![\varphi_l]\!]^{\mathcal{K}}(s) = [\![\psi_l]\!]^{\mathcal{K}}(s) < \infty$ for $l = 1, 2, 3$. From the equation for $l = 1$, we have a sequence of states $s_1, \ldots, s_m$ that satisfy $p_{\mathrm{T}}$ and that forms a list. From the equation for $l = 2$, for each $j = 1, \ldots, m$, there is a unique index $i_j$ such that $[\![\mathrm{Tile}(\alpha_{i_j}, \beta_{i_j})]\!](s_j) = 0$, and we have $1\overline{\alpha_{i_m} \cdots \alpha_{i_1}} = 1\overline{\beta_{i_m} \cdots \beta_{i_1}}$. Therefore, $\alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$. Finally, the equation for $l = 3$ guarantees $m \geq 1$.

If there is a sequence of indices $i_1, \ldots, i_m$ such that $\alpha_{i_1} \cdots \alpha_{i_m} = \beta_{i_1} \cdots \beta_{i_m}$, we can construct a Kripke structure $\mathcal{K} = (S, R, L)$ and $s \in S$ such that $[\![\varphi_i]\!]^{\mathcal{K}}(s) = [\![\psi_i]\!]^{\mathcal{K}}(s) < \infty$, as illustrated in Figure 3, where $L$ is defined as follows: for $p = p_{\mathrm{T}}$ and $p = p_{\mathrm{E}}$, $L(p, s) = 0$ if $p$ is marked in the circle for $s$ in the figure; otherwise, $L(p, s) = \infty$. For $y \in \{c, d\}$, $\xi \in \{\alpha, \beta\}$, $j \in \{1, \ldots, m\}$, and $k \in \mathbf{N}$, $L(p^y_\xi, s^k_j) = 0$ if $k \leq y(\xi_{i_j})$; otherwise, $L(p^y_x i, s^k_j) = \infty$. Using the intended meanings of the formulae, one can see that $[\![\varphi_l]\!](s_1) = [\![\psi_l]\!](s_1) < 0$ holds for $l = 1, 2, 3$. ∎

By combining Lemmas 2 and 3, we have:

**Theorem 4** Problems FORMEQ and FORMLEQ are undecidable.

**Corollary 5** The satisfiability problem of $\mathcal{L}_\rightarrow$ is undecidable.

**Proof** FORMLEQ can be reduced to the satisfiability problem of $\mathcal{L}_\rightarrow$: for given formulae $\varphi$ and $\psi$ of $\mathcal{L}$, $\varphi \rightarrow \psi$ is a formula in $\mathcal{L}_\rightarrow$ and $[\![\varphi]\!](t) \leq [\![\psi]\!](t)$ is equivalent to $[\![\varphi \rightarrow \psi]\!](t) = 0$. ∎

## 4. Decidability

In this section, we show that the satisfiability problem of $\mathcal{L}$ is decidable, namely, for given closed formula $\varphi$ of $\mathcal{L}$, we can decide whether there exists $\mathcal{K} \in \mathrm{KS}_\mathcal{L}$ for and $t \in |\mathcal{K}|$ such that $[\![\varphi]\!]^\mathcal{K}(t) = 0$. We define a translation from formulae of $\mathcal{L}$ to those of $\mathcal{L}'$ such that the given formula is satisfiable if and only if the translated formula is satisfiable. This gives a decision procedure for $\mathcal{L}$, because $\mathcal{L}'$ is decidable.

### 4.1 Translation

Figure 4 provides the formal definition of our translation; however, we will start with intuitive view.

For a given closed formula $\varphi$, we essentially define two formulae of $\mathcal{L}'$, $\mathrm{tr}(\varphi, \mathrm{Zer})$ and $\mathrm{tr}(\varphi, \mathrm{Inf})$. The former expresses that $[\![\varphi]\!](t) = 0$, the latter $[\![\varphi]\!](t) = \infty$. We introduce two more translations $\mathrm{tr}(\varphi, \mathrm{Pos})$ and $\mathrm{tr}(\varphi, \mathrm{Fin})$. They are equivalent to $\neg\mathrm{tr}(\varphi, \mathrm{Zer})$ and $\neg\mathrm{tr}(\varphi, \mathrm{Inf})$, respectively.

We consider propositional symbols $p_0$ and $p_\infty$ as expressions of $[\![p]\!](t) = 0$ and $[\![p]\!](t) = \infty$, respectively. With this in mind, the first line of Figure 4 can be read naturally. (Let us ignore the third argument $V$ of tr() presently.) The translations of $\mathbf{1}$ are also natural.

Let us skip the propositional variable $X$. The translations of negation, disjunction, and conjunction are natural from the definitions in Figure 1. For example, $[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] + [\![\varphi_2]\!] > 0$ if and only if $[\![\varphi_1]\!] > 0$ or $[\![\varphi_2]\!] > 0$.

For diamonds and boxes, two modalities introduced in $\mathcal{L}'$ play a role in distinguishing situations with finite successors from those with infinite successors. For example, consider $\mathrm{tr}(\Box\psi, \mathrm{Fin})$, which means $[\![\Box\psi]\!](t) < \infty$. It is not sufficient that all successors $t'$ of $t$ satisfy $[\![\psi]\!](t') < \infty$. In addition, the value of $[\![\psi]\!](t')$ must be zero except for finitely many successors $t'$. This observation leads to the definition of $\mathrm{tr}(\Box\psi, \mathrm{Fin}) = [1]\mathrm{tr}(\psi, \mathrm{Fin}) \wedge [\infty]\mathrm{tr}(\psi, \mathrm{Zer})$. Similar consideration should apply to $\mathrm{tr}(\Box\psi, \mathrm{Inf})$.

The first attempt of the definition of $\mathrm{tr}(\lambda X\psi, a)$ would be $\lambda X \mathrm{tr}(\psi, a)$ for $a \in$ AVL and $\lambda' X \mathrm{tr}(\psi, a)$ for $a \in$ AVG, where we define $\mu' = \nu$ and $\nu' = \mu$.

$$
\begin{array}{l}
\mathrm{tr}(p, \mathrm{Zer}, V) = p_0 \wedge \neg p_\infty, \quad \mathrm{tr}(p, \mathrm{Pos}, V) = \neg p_0 \vee p_\infty \\
\mathrm{tr}(p, \mathrm{Inf}, V) = \neg p_0 \wedge p_\infty, \quad \mathrm{tr}(p, \mathrm{Fin}, V) = p_0 \vee \neg p_\infty \\
\mathrm{tr}(\mathbf{1}, \mathrm{Zer}, V) = \mathrm{tr}(\mathbf{1}, \mathrm{Inf}, V) = \mathbf{false} \\
\mathrm{tr}(\mathbf{1}, \mathrm{Fin}, V) = \mathrm{tr}(\mathbf{1}, \mathrm{Pos}, V) = \mathbf{true} \\
\mathrm{tr}(X, a, V) = \mathrm{tr}(\mathrm{BF}(X), a, V) \\
\mathrm{tr}(\neg\psi, \mathrm{Zer}, V) = \mathrm{tr}(\neg\psi, \mathrm{Fin}, V) = \mathrm{tr}(\psi, \mathrm{Inf}, V[(X, a) \mapsto 3 \mid V(X, a) = 2]) \\
\mathrm{tr}(\neg\psi, \mathrm{Pos}, V) = \mathrm{tr}(\neg\psi, \mathrm{Inf}, V) = \mathrm{tr}(\psi, \mathrm{Fin}, V[(X, a) \mapsto 3 \mid V(X, a) = 2]) \\
\mathrm{tr}(\psi_1 \vee \psi_2, a, V) = \begin{cases} \mathrm{tr}(\psi_1, a, V) \vee \mathrm{tr}(\psi_2, a, V) & \text{if } a \in \mathrm{AVL} \\ \mathrm{tr}(\psi_1, a, V) \wedge \mathrm{tr}(\psi_2, a, V) & \text{if } a \in \mathrm{AVG} \end{cases} \\
\mathrm{tr}(\psi_1 \wedge \psi_2, a, V) = \begin{cases} \mathrm{tr}(\psi_1, a, V) \wedge \mathrm{tr}(\psi_2, a, V) & \text{if } a \in \mathrm{AVL} \\ \mathrm{tr}(\psi_1, a, V) \vee \mathrm{tr}(\psi_2, a, V) & \text{if } a \in \mathrm{AVG} \end{cases} \\
\mathrm{tr}(\Diamond\psi, a, V) = \begin{cases} \langle 1\rangle\mathrm{tr}(\psi, a, V) \vee \langle\infty\rangle\mathrm{tr}(\psi, a, V) & \text{if } a \in \mathrm{AVL} \\ [1]\mathrm{tr}(\psi, a, V) \wedge [\infty]\mathrm{tr}(\psi, a, V) & \text{if } a \in \mathrm{AVG} \end{cases} \\
\mathrm{tr}(\Box\psi, a, V) = \begin{cases} [1]\mathrm{tr}(\psi, a, V) \wedge [\infty]\mathrm{tr}(\psi, \mathrm{Zer}, V) & \text{if } a \in \mathrm{AVL} \\ \langle 1\rangle\mathrm{tr}(\psi, a, V) \vee \langle\infty\rangle\mathrm{tr}(\psi, \mathrm{Pos}, V) & \text{if } a \in \mathrm{AVG} \end{cases} \\
\mathrm{tr}(\lambda X\psi, a, V) = \begin{cases} X_a & \text{if } V(X, a) = 1 \text{ or } V(X, a) = 2 \\ X_{\mathrm{neg}} & \text{if } V(X, a) = 3 \\ \text{the following are applied} & \text{if } V(X, a) = 0 \end{cases} \\
\mathrm{tr}(\mu X\psi, a, V) = \begin{cases} \mu X_a\ \mathrm{tr}(\psi, a, V'[(X, a) \mapsto 1]) & \text{if } a \in \mathrm{AVL} \\ \nu X_a\ \mathrm{tr}(\psi, a, V'[(X, a) \mapsto 1]) & \text{if } a \in \mathrm{AVG} \end{cases} \\
\mathrm{tr}(\nu X\psi, a, V) = \begin{cases} \nu X_a\ \mathrm{tr}(\psi, a, V'[(X, a) \mapsto 1]) & \text{if } a = \mathrm{Zer} \\ \mu X_a\ \mathrm{tr}(\psi, a, V'[(X.a) \mapsto 1]) & \text{if } a = \mathrm{Pos} \end{cases} \\
\mathrm{tr}(\nu X\psi, \mathrm{Fin}, V) = \\
\qquad \nu X_{\mathrm{neg}} \mu X_{\mathrm{Fin}}\ (\mathrm{tr}(\psi, \mathrm{Fin}, V'[(X, \mathrm{Fin}) \mapsto 2]) \vee \mathrm{tr}(\nu X\psi, \mathrm{Zer}, V'[(X, \mathrm{Fin}) \mapsto 2])) \\
\mathrm{tr}(\nu X\psi, \mathrm{Inf}, V) = \\
\qquad \mu X_{\mathrm{neg}} \nu X_{\mathrm{Inf}}\ (\mathrm{tr}(\psi, \mathrm{Inf}, V'[(X, \mathrm{Inf}) \mapsto 2]) \wedge \mathrm{tr}(\nu X\psi, \mathrm{Pos}, V'[(X, \mathrm{Inf}) \mapsto 2])) \\
\hfill \text{where } V' = V[(Y, a) \mapsto 0 \mid Y \prec X, a \in \mathrm{AV}]
\end{array}
$$

**Fig. 4** Translation

One problem is that the value $a$ may be changed during the recursion. For example, consider $\mathrm{tr}(\varphi, \mathrm{Zer})$ for $\varphi = \mu X(\Box X)$. The translation becomes $\mu X(\langle 1\rangle \mathrm{tr}(X, \mathrm{Zer}) \vee \langle\infty\rangle \mathrm{tr}(X, \mathrm{Pos}))$. It is all right to replace $\mathrm{tr}(X, \mathrm{Zer})$ with $X$, but $\mathrm{tr}(X, \mathrm{Pos})$ cannot be replaced with $X$. Instead, it should be replaced with $\mathrm{tr}(\varphi, \mathrm{Pos})$. To achieve this, we introduce the third parameter $V$ of tr(). $V$ is a function from $\mathrm{PV} \times \mathrm{AV}$ to $\{0, 1, 2, 3\}$. Let us denote the set of such functions by $\mathcal{V}$. $V$ records binding of variables in the following manner. $V$ is initialized by $V_{\mathrm{I}}$, which is defined by $V_{\mathrm{I}}(X, a) = 0$ for all $X$ and $a$. When $\mathrm{tr}(\lambda X\psi, a, V)$ is processed, $V(X, a)$ is set to 1. Moreover, $V(Y, \cdot)$ is reset to 0 for all proper subformulae $Y$ of $X$, to ensure that the $\prec$-order of the translated formula is consistent with that of the original formula. At a later stage, when $\mathrm{tr}(X, a, V)$ is processed, it is replaced with $X_a$ if $V(X, a) = 1$, else the translation continues with $\mathrm{tr}(\mathrm{BF}(X), a, V)$.

Another and the most essential problem is how to handle cases $\mathrm{tr}(\nu X\psi, \mathrm{Fin}, V)$ and $\mathrm{tr}(\nu X\psi, \mathrm{Inf}, V)$. We only examine the former, because the latter is merely its negation. As an example, let $\varphi_1 = \nu X(p \wedge \Diamond X)$ and $\mathcal{K}_1 = (T, R, L)$ be the Kripke structure defined by $T = \{t_n \mid n \in \mathbf{N}\}$, $R = \{(t_n, t_{n+1}) \mid n \in \mathbf{N}\}$, and $L(t_n) = 1$ for all $n \in \mathbf{N}$. The corresponding Kripke structure $\mathcal{K}_1'$ for $\mathcal{L}'$ is, although we have not yet defined it, $(\mathbf{N}, R', L')$, where $R'(1) = R$, $R'(\infty) = \varnothing$, and $L'(p_0) = L'(p_\infty) = \varnothing$.

Let $\varphi_1'$ be the result of the naive translation of $\mathrm{tr}(\varphi_1, \mathrm{Fin})$: $\varphi_1' = \nu X((p_0 \vee \neg p_\infty) \wedge ([1]X \vee [\infty]X))$. We have $[\![\varphi_1]\!](t_0) = \infty$ but $t_0 \models \varphi_1'$. Here, $[\![\varphi_1]\!](t_0)$ can be calculated by "developing" $\varphi_1$, $[\![\varphi_1]\!](t_0) = L(p, t_0) + [\![\varphi_1]\!](t_1) = L(p, t_0) + L(p, t_1) + [\![\varphi_1]\!](t_2) = \cdots$. This observation suggests that the value $[\![\varphi_1]\!](t)$ must become zero after the formula is developed finitely many times, if $[\![\varphi_1]\!](t) < \infty$. Thus, we have an improved (but still incorrect) definition: $\mathrm{tr}_1(\nu X\psi, \mathrm{Fin}, V) = \mu X_{\mathrm{Fin}}(\mathrm{tr}(\psi, \mathrm{Fin}, V') \vee \mathrm{tr}(\nu X\psi, \mathrm{Zer}, V'))$, for some appropriately specified $V'$.

The remaining problem is caused by the negation. For example, let $\varphi_2 = \nu X(p \wedge \Diamond\neg\neg X)$. In this case, $[\![\varphi_2]\!]^{\mathcal{K}_1}(t_0) = 1 < \infty$ holds, but $\mathcal{K}_1', t_0 \not\models \mathrm{tr}_1(\varphi_2, \mathrm{Fin})$. A double negation "resets" the value to 0. Therefore, if a development path passes negation symbols for infinitely many times, the value remains finite. To realize this observation, we introduce another propositional variable $X_{\mathrm{neg}}$ and put $\nu X_{\mathrm{neg}}$ at the beginning of the translated formula. When the variable $X$ is processed during the translation, we replace it with $X_{\mathrm{neg}}$ if $X$ is in the scope of a negation operator; otherwise, we use $X_{\mathrm{Fin}}$, which is bound by the $\mu$ operator. To realize this, we first set $V(X, \mathrm{Fin})$ to 2, and when it encounters a negation symbol, the value is changed to 3.

The detailed definition of $\mathrm{tr}(\psi, a, V)$ is given in Figure 4. We define $\mathrm{tr}(\psi, a) = \mathrm{tr}(\psi, a, V_{\mathrm{I}})$. The translation process always terminates, because the value of $V(X, a)$ for $\prec$-larger variable continuously becomes non-zero. For details, refer to Lemma 15 in Appendix.

We conclude this section by stating a lemma, which can be proved by checking the definition. Let $\varphi$ be a formula in $\mathcal{L}$ and $a \in \mathrm{AV}$. For propositional variable $X$ in $\varphi$, the set of propositional variables in the form of $X_{\mathrm{neg}}$ or $X_b$ for some $b \in \mathrm{AV}$ appearing in $\varphi' = \mathrm{tr}(\varphi, a)$ is denoted by $C(X)$. Such a propositional variable can be bound two or more times in $\chi$. We consider variables bound by different fixed-point operators are different variables.

**Lemma 6**

(1) $\mathrm{tr}(\neg\varphi, a, V) \equiv \neg\mathrm{tr}(\varphi, a, V)$.

(2) $\mathrm{tr}(\varphi, \mathrm{Pos}, V) \equiv \neg\mathrm{tr}(\varphi, \mathrm{Zer}, V)$ and $\mathrm{tr}(\varphi, \mathrm{Fin}, V) \equiv \neg\mathrm{tr}(\varphi, \mathrm{Inf}, V)$.

(3) If $X$ and $Y$ are propositional variables in $\varphi$, $Y \prec X$, $Y' \in C(Y)$, and $X' \in C(X)$, then $Y'$ does not occur freely in $\mathrm{BF}(X')$.

### 4.2 Outline of Equi-satisfiability Proof

Let $\varphi_{\mathrm{I}}$ be a closed formula in $\mathcal{L}$ and $a_{\mathrm{I}} \in \mathrm{AV}$. The following is the main theorem of Section 4.

**Theorem 7** The following are equivalent.

(1) $\varphi_{\mathrm{I}}$ is $a_{\mathrm{I}}$-satisfiable.

(2) $\mathrm{tr}(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$ is satisfiable.

To prove this theorem, we introduce a relation between $(\mathcal{K}, t)$ and $(\mathcal{K}', s)$ where $\mathcal{K} \in \mathrm{KS}_{\mathcal{L}}$, $t \in |\mathcal{K}|$, $\mathcal{K}' \in \mathrm{KS}_{\mathcal{L}'}$, and $s \in |\mathcal{K}'|$, called a $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-simulation (Section 4.5). If this relation exists, $\varphi_{\mathrm{I}}$ and $\mathrm{tr}(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$ behaves similarly on $\mathcal{K}$ and $\mathcal{K}'$, respectively. In particular, we prove the following lemma (Section 4.6).

**Lemma 8** Let $\mathcal{K} \in \mathrm{KS}_{\mathcal{L}}$, $\mathcal{K}' \in \mathrm{KS}_{\mathcal{L}'}$, $t_{\mathrm{I}} \in |\mathcal{K}|$, $s_{\mathrm{I}} \in |\mathcal{K}'|$, and $a_{\mathrm{I}} \in \mathrm{AV}$. If a $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-simulation between $(\mathcal{K}, t_{\mathrm{I}})$ and $(\mathcal{K}', s_{\mathrm{I}})$ exists, then we have

$$[\![\varphi]\!]^{\mathcal{K}}(t_{\mathrm{I}}) \in \gamma(a_{\mathrm{I}}) \iff \mathcal{K}', s_{\mathrm{I}} \models \mathrm{tr}(\varphi_{\mathrm{I}}, a_{\mathrm{I}}).$$

Therefore, the following two lemmas are sufficient to prove Theorem 7.

**Lemma 9** If $\varphi_{\mathrm{I}}$ is $a_{\mathrm{I}}$-satisfiable, then there exist $\mathcal{K} \in \mathrm{KS}_{\mathcal{L}}$, $\mathcal{K}' \in \mathrm{KS}_{\mathcal{L}'}$, $t_{\mathrm{I}} \in |\mathcal{K}|$, and $s_{\mathrm{I}} \in |\mathcal{K}'|$ such that $[\![\varphi_{\mathrm{I}}]\!]^{\mathcal{K}}(t_{\mathrm{I}}) \in \gamma(a_{\mathrm{I}})$ and there is a $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-simulation between $(\mathcal{K}, t_{\mathrm{I}})$ and $(\mathcal{K}', s_{\mathrm{I}})$.

**Lemma 10** If closed formula $\psi$ of $\mathcal{L}'$ is satisfiable, then there exist $\mathcal{K} \in \mathrm{KS}_{\mathcal{L}}$, $\mathcal{K}' \in \mathrm{KS}_{\mathcal{L}'}$, $t_{\mathrm{I}} \in |\mathcal{K}|$, and $s_{\mathrm{I}} \in |\mathcal{K}'|$ such that $\mathcal{K}', s_{\mathrm{I}} \models \psi$ and there is a $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-simulation between $(\mathcal{K}, t_{\mathrm{I}})$ and $(\mathcal{K}', s_{\mathrm{I}})$.

### 4.3 Intermediate Interpretation

Proving Lemma 8 using induction on the construction of $\varphi$ is apparently difficult from the nature of the translation. Instead, we plan to use the game expression for $\mathcal{K}'$ reviewed in Section 2.2. We need some mechanism that can be used on $\mathcal{K}$ as a counterpart of the game on $\mathcal{K}'$. For this purpose, we use sequences of ordinal numbers, which show the indices of $F_\alpha$ used in the definition of $[\![\varphi]\!]$ (Figure 1). As there are two or more fixed-point operators that appear in a formula, we need sequences of ordinal numbers.

For $\varphi \in \mathrm{SF}(\varphi_{\mathrm{I}})$, we define $\mathrm{FV}_\varphi = \{X \in \mathrm{PV} \mid \varphi < \mathrm{BF}(X)\}$ and $\mathrm{fv}(\varphi) = |\mathrm{FV}_\varphi|$. For $X \in \mathrm{PV}$, we denote $\mathrm{fv}(\mathrm{BF}(X))$ by $\mathrm{idx}(X)$. For example, let $\varphi_{\mathrm{I}} = \mu X(p \vee \nu Y \nu Z(X \wedge Y \wedge Z \wedge q))$. Then, $\mathrm{FV}_p = \{X\}$, $\mathrm{FV}_q = \{X, Y, Z\}$, $\mathrm{fv}(p) = 1$, $\mathrm{fv}(q) = 3$, $\mathrm{idx}(X) = 0$, $\mathrm{idx}(Y) = 1$, and $\mathrm{idx}(Z) = 2$. Thus, $\mathrm{idx}(X)$ is the nesting depth of its binding fixed-point operator.

Let Seq be the class of finite sequences of ordinal numbers, $\mathrm{Seq}_l = \{\xi \in \mathrm{Seq} \mid \mathrm{len}(\xi) = l\}$ for $l \in \mathbf{N}$, and $\mathrm{Seq}_\varphi = \mathrm{Seq}_{\mathrm{fv}(\varphi)}$ for $\varphi \in \mathrm{SF}(\varphi_{\mathrm{I}})$. An ordinal number $\alpha$ is considered as a sequence of length one, i.e., an element of $\mathrm{Seq}_1$. For $\xi \in \mathrm{Seq}$, its $l$-th element is denoted by $\xi(l)$. For $\xi, \xi' \in \mathrm{Seq}$, their concatenation is denoted by $\xi : \xi'$. If $n \geq \mathrm{len}(\xi)$, the prefix of $\xi$ with length $n$ is denoted by $\xi \upharpoonright n$.

Let $\mathcal{K} = (T, R, L)$ be a Kripke structure for $\mathcal{L}$. For $\varphi \in \mathrm{SF}(\varphi_{\mathrm{I}})$, $\xi \in \mathrm{Seq}_\varphi$, and $t \in T$, we define $\langle\varphi\rangle_\xi(t) \in \mathbf{N}_\infty$ as in Figure 5. It is the value $[\![\varphi]\!](t)$ at its $\xi$'th iteration. For example, if we use $\varphi_{\mathrm{I}}$ mentioned above and let $\varphi = X \wedge Y \wedge Z \wedge q$, then $\langle\varphi\rangle_{\alpha:\beta:\gamma}(t)$ is the value of $[\![\varphi]\!]^\rho(t)$ at $\alpha$'th iteration for $X$, $\beta$'th iteration for $Y$, and $\gamma$'th iteration for $Z$.

Let $\varphi \in \mathrm{SF}(\varphi_{\mathrm{I}})$ and $\xi \in \mathrm{Seq}$. If $\mathrm{idx}(X) < \mathrm{len}(\xi)$ holds for all propositional variable $X$ that freely occurs in $\varphi$, then for any extension $\xi' \in \mathrm{Seq}_\varphi$ of $\xi$, the value of $\langle\varphi\rangle_{\xi'}$ is identical. We denote this value by $\langle\varphi\rangle_\xi$. This notation is frequently used especially when $\varphi = X$ and $\mathrm{len}(\xi) = \mathrm{idx}(X) + 1$.

$$\langle p\rangle_\xi(t) = L(p, t), \quad \langle \mathbf{1}\rangle_\xi(t) = 1$$
$$\langle X\rangle_{\xi:\alpha:\xi'}(t) = \begin{cases} \inf\{\langle \mathrm{BFS}(X)\rangle_{\xi:\beta}(t) \mid \beta < \alpha\} & \text{if } \lambda_X = \mu \\ \sup\{\langle \mathrm{BFS}(X)\rangle_{\xi:\beta}(t) \mid \beta < \alpha\} & \text{if } \lambda_X = \nu \end{cases} \quad \text{where } \mathrm{len}(\xi) = \mathrm{idx}(X)$$
$$\langle\neg\psi\rangle_\xi(t) = \begin{cases} 0 & \text{if } \langle\psi\rangle_\xi(t) = \infty \\ \infty & \text{if } \langle\psi\rangle_\xi(t) < \infty \end{cases}$$
$$\langle\psi_1 \vee \psi_2\rangle_\xi(t) = \min(\langle\psi_1\rangle_\xi(t), \langle\psi_2\rangle_\xi(t))$$
$$\langle\psi_1 \wedge \psi_2\rangle_\xi(t) = \langle\psi_1\rangle_\xi(t) + \langle\psi_2\rangle_\xi(t)$$
$$\langle\Diamond\psi\rangle_\xi(t) = \min(\langle\psi\rangle_\xi(t') \mid (t, t') \in R)$$
$$\langle\Box\psi\rangle_\xi(t) = \textstyle\sum(\langle\psi\rangle_\xi(t') \mid (t, t') \in R)$$
$$\langle\mu X\psi\rangle_\xi(t) = \inf\{\langle\psi\rangle_{\xi:\alpha}(t) \mid \alpha \in \mathrm{On}\}$$
$$\langle\nu X\psi\rangle_\xi(t) = \sup\{\langle\psi\rangle_{\xi:\alpha}(t) \mid \alpha \in \mathrm{On}\}$$

**Fig. 5** Definition of $\langle\varphi\rangle_\xi(t)$

The next lemma describes a relation between $[\![\varphi]\!]$ and $\langle\varphi\rangle_\xi$. For a proof, refer to Appendix.

**Lemma 11**

(1) Assume that $\varphi \in \mathrm{SF}(\varphi_{\mathrm{I}})$, $\xi \in \mathrm{Seq}_\varphi$, and $\rho$ is a valuation such that $\rho(X) = \langle X\rangle_\xi$ for any $X \in \mathrm{FV}_\varphi$. Then, $[\![\varphi]\!]^\rho = \langle\varphi\rangle_\xi$.

(2) $[\![\varphi_{\mathrm{I}}]\!] = \langle\varphi_{\mathrm{I}}\rangle_\epsilon$, where $\epsilon$ is the empty sequence.

(3) There is an ordinal number $\kappa'$ such that for any ordinal number $\kappa \geq \kappa'$, $\lambda X\psi \in \mathrm{SF}(\varphi_{\mathrm{I}})$, $t \in T$, and $\xi \in \mathrm{Seq}_\varphi$, the following holds.
$$\langle\lambda X\psi\rangle_\xi(t) = \langle\psi\rangle_{\xi:\kappa}(t) = \langle X\rangle_{\xi:\kappa}(t).$$

We fix an ordinal number $\kappa'$ that satisfies Lemma 11 (3) and let $\kappa = \kappa' + 1$. Hereafter in this paper, $\kappa$ refers to this ordinal number.

### 4.4 $W$-sequence

Although we will use the game expression of ordinary semantics on $\mathcal{K}'$, we do not formally define games or strategies on $\mathcal{K}$. Instead, we introduce sequences of quadruple, called $W$-sequence. Intuitively, a $W$-sequence represents a "play" of the game in which Player obeys a standard "strategy."

For $\varphi \in \mathrm{SF}(\varphi_{\mathrm{I}})$ and $l < \mathrm{fv}(\varphi)$, let $X(\varphi, l)$ be the propositional variable $X \in \mathrm{FV}_\varphi$ such that $\mathrm{idx}(X) = l$.

Let $\mathcal{K} = (T, R, L)$ be a Kripke structure for $\mathcal{L}$, $\varphi \in \mathrm{SF}(\varphi_{\mathrm{I}})$, $\xi \in \mathrm{Seq}_\varphi$, $\alpha \in \mathrm{On}$, and $t \in T$. We define $\mathrm{NuLim}(\xi, \varphi)$ as the set of natural numbers $l < \mathrm{fv}(\varphi)$ that satisfy the following conditions: (1) $\varphi$ is positive in $\mathrm{BF}(X)$, where $X = X(\varphi, l)$, (2) $\lambda_X = \nu$, (3) $\xi(l)$ is a limit ordinal number, and (4) $\xi(l) < \kappa$. When $\mathrm{NuLim}(\xi, \varphi) \neq \varnothing$, we denote its least element by $\mathrm{pni}(\xi, \varphi)$ (principal nu-limit index). If $\mathrm{NuLim}(\xi, \varphi) = \varnothing$, we define $\mathrm{pni}(\xi, \varphi) = -1$. For $l < \mathrm{fv}(\varphi)$, we denote by $\xi\{l \mapsto \alpha\}$ sequence $\eta \in \mathrm{Seq}_\varphi$ defined by $\eta \restriction l = \xi \restriction l$, $\eta(l) = \alpha$, and $\eta(l') = \kappa$ for $l' > l$. The value of $\varphi$ at $t$ is *strictly continuous* on $\xi$, written by $\mathrm{SCont}(\varphi, \xi, t)$, if

- $l = \mathrm{pni}(\xi, \varphi) \geq 0$,
- $\langle \varphi \rangle_\xi(t) = \sup\{\langle \varphi \rangle_{\xi\{l \mapsto \alpha\}}(t) \mid \alpha < \xi(l)\} = \infty$, and
- $\langle \varphi \rangle_{\xi\{l \mapsto \alpha\}}(t) < \infty$ for all $\alpha < \xi(l)$.

For $t \in T$, we define $\mathrm{Suc}(t) = \{t' \in T \mid (t, t') \in R\}$.

The information we need to keep track of is the member of the following set:

$$W = \{(\varphi, \xi, t, a) \in \mathrm{SF}(\varphi_\mathrm{I}) \times \mathrm{Seq} \times T \times \mathrm{AV} \mid \xi \in \mathrm{Seq}_\varphi \text{ and } \langle \varphi \rangle_\xi(t) \in \gamma(a)\}.$$

A relation $R^W$ on $W$ is the set of pairs $(w, w') \in R^W$ that satisfy one of the conditions in Figure 6, where $w = (\varphi, \xi, t, a)$ and $w' = (\varphi', \xi', t', a')$. Condition $\mathrm{FP}(\lambda, \hat{\xi})$ that appears in the figure is defined in Figure 7. Intuitively, the relation consists of the "move of Player that obeys the standard strategy" and all possible "moves of Opponent."

For $w \in W$, the set $\{w' \in W \mid (w, w') \in R^W\}$ is also denoted by $\mathrm{Suc}(w)$. For $w \in W$, we write $w = (w.\varphi, w.\xi, w.t, w.a)$.

A finite or infinite sequence $w = (w_i)_i$ of $W$ is a *$W$-sequence* if $(w_i, w_{i+1}) \in R^W$ for each $i$ such that $w_i$ and $w_{i+1}$ are defined.

For any infinite $W$-sequence $w$, it is clear that there is $X \in \mathrm{PV}$ such that $w(i).\varphi = X$ for infinitely many $i$. We call the $\prec$-largest such propositional variable the *principal variable* of the sequence.

Let $w$ be a $W$-sequence. It is clear that for each propositional variable $X$, $\{w(i).a \mid w(i).\varphi = X\}$ is either a subset of AVL or a subset of AVG. If it is a non-empty subset of AVL, we call $X$ an AVL variable. If it is a non-empty subset of AVG, we call $X$ an AVG variable. The set of AVL (AVG, resp.) variables is denoted by $\mathrm{PV_L}$ ($\mathrm{PV_G}$, resp.).

- $\varphi = \neg\varphi'$, $\xi = \xi'$, $t = t'$, and either $a \in \mathrm{AVL}$ and $a' = \mathrm{Inf}$ or $a \in \mathrm{AVG}$ and $a' = \mathrm{Fin}$.
- Either of the following two
  - $\varphi = \varphi' \vee \psi$ or $\varphi = \psi \vee \varphi'$, $t = t'$, $a = a'$
  - $\varphi = \Diamond\varphi'$, $(t, t') \in R$, $a = a'$

  and both of the following:
  - If $a = \mathrm{Inf}$, $l = \mathrm{pni}(\xi, \varphi) \geq 0$, and $\alpha_0 = \min\{\alpha \mid \langle \varphi' \rangle_{\xi\{l \mapsto \alpha\}}(t') = \infty\} < \xi(l)$, then $\xi' = \xi\{l \mapsto \alpha_0\}$; otherwise, $\xi' = \xi$.
  - If $a \in \mathrm{AVL}$, then $\langle \varphi \rangle_\xi(t) = \langle \varphi' \rangle_{\xi'}(t')$.
- $\varphi = \varphi' \wedge \psi$ or $\varphi = \psi \wedge \varphi'$, $\xi = \xi'$, $t = t'$, and $a = a'$. Furthermore, if $a = \mathrm{Inf}$ and $\mathrm{SCont}(\varphi, \xi, t)$, then $\mathrm{SCont}(\varphi', \xi', t')$.
- $\varphi = \Box\varphi'$, $(t, t') \in R$, either $a = a'$, $(a, a') = (\mathrm{Inf}, \mathrm{Pos})$, or $(a, a') = (\mathrm{Fin}, \mathrm{Zer})$, and:
  - If $a \neq \mathrm{Inf}$, then $\xi = \xi'$.
  - If $a = \mathrm{Inf}$, then
    - If $\mathrm{SCont}(\varphi, \xi, t)$ holds, then
      - If there is $t'' \in \mathrm{Suc}(t)$ such that $\mathrm{SCont}(\varphi', \xi, t'')$ and $\langle \varphi' \rangle_\xi(t'') = \infty$, then $a' = \mathrm{Inf}$, $\xi = \xi'$, and $\mathrm{SCont}(\varphi', \xi, t')$.
      - Otherwise, $a' = \mathrm{Pos}$ and $\xi' = \xi\{l \mapsto \alpha_0\}$, where $l = \mathrm{pni}(\xi, \varphi)$ and $\alpha_0 = \min\{\alpha \mid \langle \varphi' \rangle_{\xi\{l \mapsto \alpha\}}(t') > 0\}$. Note that in this case we have $\alpha_0 < \xi(l)$ because $\mathrm{SCont}(\varphi, \xi, t)$ holds.
    - Otherwise,
      - $\xi = \xi'$.
      - If there is $t'' \in \mathrm{Suc}(t)$ such that $\langle \varphi' \rangle_\xi(t'') = \infty$, then $a' = \mathrm{Inf}$.
      - Otherwise, $a' = \mathrm{Pos}$.
- $\varphi = \lambda X \varphi'$ and $\mathrm{FP}(\lambda, \xi)$.
- $\varphi = X$, $\varphi' = \mathrm{BFS}(X)$, and $\mathrm{FP}(\lambda_X, \xi \restriction \mathrm{idx}(X))$.

**Fig. 6** Relation $R^W$

**Lemma 12** Let $((\varphi_i, \xi_i, t_i, a_i) \mid i \in \mathbf{N})$ be an infinite $W$-sequence, $X$ be its principal variable, and $l = \mathrm{idx}(X)$.

(1) there is $i_0 \in \mathbf{N}$ such that for any $i \geq i_0$, $\xi_i \restriction (l+1) = \xi_{i_0} \restriction (l+1)$.

(2) $X \notin \mathrm{PV}_\mu \cap \mathrm{PV_L}$.

(3) If $X \in \mathrm{PV}_\nu \cap \mathrm{PV_G}$, then, there is $i_0$ such that for all $i \geq i_0$, $\varphi_i = X$ implies
- $a_{i+1} = \mathrm{Inf}$
- $\alpha = \xi_i(l)$ is a limit ordinal number.
- $\langle X \rangle_{(\xi_i \restriction l):\beta}(t_i) < \infty$ for all $\beta < \alpha$.

(4) If $\lambda_X = \nu$ and $X$ is an AVG-variable, then there are only finitely many $i$ such that $\varphi_i = \neg\varphi_{i+1}$.

- $t = t'$
- If either $\lambda = \mu$, $a = \mathrm{Zer}$, or $a = \mathrm{Pos}$, then $a = a'$.
- If $\lambda = \nu$ and $a = \mathrm{Inf}$, then $a' = \mathrm{Inf}$ or $a' = \mathrm{Pos}$.
- If $\lambda = \nu$, $a = \mathrm{Fin}$, and $\langle\varphi\rangle_{\hat{\xi}}(t) > 0$, then $a' = \mathrm{Fin}$.
- If $\lambda = \nu$, $a = \mathrm{Fin}$, and $\langle\varphi\rangle_{\hat{\xi}}(t) = 0$, then $a' = \mathrm{Zer}$.
- If either $\lambda = \mu$ and $a \in \mathrm{AVG}$ or $\lambda = \nu$ and $a \in \mathrm{AVL}$, then $\xi' = \hat{\xi} : \kappa$.
- If $\lambda = \mu$ and $a \in \mathrm{AVL}$, then $\xi' = \hat{\xi} : \alpha$,
  where $\alpha = \min\{\alpha \mid \langle\varphi'\rangle_{\hat{\xi}:\alpha}(t') = \langle\varphi\rangle_{\hat{\xi}}(t)\}$.
- If $\lambda = \nu$ and $a \in \mathrm{AVG}$,
  - If $(a, a') = (\mathrm{Inf}, \mathrm{Pos})$ and $0 \leq l = \mathrm{pni}(\hat{\xi}, \varphi) \leq \mathrm{idx}(X)$, then $\xi' = \hat{\xi}\{l \mapsto \alpha\}$, where $\alpha = \min\{\alpha \mid \langle\varphi'\rangle_{\hat{\xi}\{l\mapsto\alpha\}}(t) \in \gamma(a')\}$.
  - Otherwise, $\xi' = \hat{\xi} : \alpha$,
    where $\alpha = \min\{\alpha \mid \langle\varphi'\rangle_{\hat{\xi}:\alpha}(t') \in \gamma(a')\}$.

**Fig. 7** Condition $\mathrm{FP}(\lambda, \hat{\xi})$

(5) If $\lambda_X = \nu$, $X$ is an AVL variable, and there are only finitely many $i$ such that $\varphi_i = \neg\varphi_{i+1}$, then there are only finitely many $i$ such that $\langle\varphi_i\rangle_{\xi_i}(t_i) > 0$.

For a proof, refer to Appendix. This lemma can be considered as an $\mathbf{N}_\infty$-semantics version of Theorem 1. In fact, if we define $W$-sequence with ordinary semantics in a similar way, Theorem 1 corresponds to (1) + (2) + (3′), where (3′) is a simple statement "$X \notin \mathrm{PV}_\nu \cap \mathrm{PV}_{\mathrm{G}}$."

### 4.5 $(\varphi, a)$-simulation

In this section, we define $(\varphi, a)$-simulation. Note that $\mathcal{L}'$ has two modalities 1 and $\infty$. We need to take care so that a transition labeled with $\infty$ in $\mathcal{K}'$ should correspond to infinitely many transitions in $\mathcal{K}$, while a transition labeled with 1 in $\mathcal{K}'$ should correspond to finitely many transitions in $\mathcal{K}$. This consideration leads to the following definition.

Let $t_{\mathrm{I}} \in T$, $s_{\mathrm{I}} \in S$, and $a_{\mathrm{I}} \in \mathrm{AV}$. A relation $Q \subseteq T \times S$ is a $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-*simulation* between $(\mathcal{K}, t_{\mathrm{I}})$ and $(\mathcal{K}', s_{\mathrm{I}})$ if the following conditions are satisfied.

(1) $(t_{\mathrm{I}}, s_{\mathrm{I}}) \in Q$.

(2) If $(t, s) \in Q$, then for $p \in \mathrm{PS}$,
- $L(p, t) = 0 \iff s \in L'(p_0) \setminus L'(p_\infty)$
- $L(p, t) = \infty \iff s \in L'(p_\infty) \setminus L'(p_0)$

(3) If $(t, s), (t', s') \in Q$, then $(t, t') \in R \iff (s, s') \in R'(1) \cup R'(\infty)$.

(4) If $(t, s) \in Q$ and $w = (\varphi, \xi, t, a)$ is a descendant of $w_{\mathrm{I}} = (\varphi_{\mathrm{I}}, \epsilon, t_{\mathrm{I}}, a_{\mathrm{I}})$; i.e., if there is a finite $W$-sequence $(w_i \mid 0 \leq i \leq i_0)$ such that $w_0 = w_{\mathrm{I}}$ and $w_{i_0} = w$; then the following are satisfied:
- If either $\varphi = \Diamond\psi$ and $a \in \mathrm{AVL}$ or $\varphi = \Box\psi$ and $a \in \mathrm{AVG}$, then there exist $w' \in W$ and $s' \in S$ such that $(w'.t, s') \in Q$, $(s, s') \in R'(1) \cup R'(\infty)$, $(w, w') \in R^W$, and if $(a, w'.a) = (\mathrm{Inf}, \mathrm{Pos})$ then $(s, s') \in R'(\infty)$.
- If either $\varphi = \Diamond\psi$ and $a \in \mathrm{AVG}$ or $\varphi = \Box\psi$ and $a \in \mathrm{AVL}$, then for each $s' \in S$ such that $(s, s') \in R'(1) \cup R'(\infty)$, there exists $w' \in W$ such that $(w'.t, s') \in Q$ and $(w, w') \in R^W$. Moreover, if $(s, s') \in R'(\infty)$ and $a = \mathrm{Fin}$, then we can take $w'$ so that $w'.a = \mathrm{Zer}$.

With this definition, we can prove Lemmas 9 and 10. For a proof, refer to Appendix.

### 4.6 Strategy

Assume that $Q$ is a $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-simulation between $(\mathcal{K}, t_{\mathrm{I}})$ and $(\mathcal{K}', s_{\mathrm{I}})$, and $[\![\varphi_{\mathrm{I}}]\!]^{\mathcal{K}}(t_{\mathrm{I}}) \in \gamma(a_{\mathrm{I}})$.

The following lemma can be proved using the results obtained so far. For a detailed proof, refer to Appendix.

**Lemma 13** There are a strategy $\sigma$ for Player in the $\mu$-calculus game at $(\mathrm{tr}(\varphi_{\mathrm{I}}, a_{\mathrm{I}}), s_{\mathrm{I}})$ and a $W$-sequence $(w_i)_i$ that satisfy the following conditions. We write $w_i = (\varphi_i, \xi_i, t_i, a_i)$.

(1) If Player obeys $\sigma$, for each index $i$ of the $W$-sequence, $(\mathrm{tr}(\varphi_i, a_i, V), s_i)$ (for some $V \in \mathcal{V}$) appears as a position in the play. Moreover, if the play is finite, the $W$ sequence is finite and the last $\varphi_i$ is either a propositional symbol or $\mathbf{1}$.

(2) $(t_i, s_i) \in Q$ for each index $i$.

**Lemma 14** Strategy $\sigma$ is a winning strategy for Player.

**Proof** If the play is finite, Player wins because $w_i \in W$. Therefore we can assume that the play is infinite. Thus, $(w_i)_{i\in\mathbf{N}}$ is an infinite $W$-sequence. Let $X$ be the principal variable of the sequence, and $X'$ be the $\prec$-largest propositional variable in $\mathrm{tr}(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$ that appears infinitely often in the play. By Lemma 6 (3), $X' \in C(X)$. What we need to show is that $X'$ is a $\nu$-variable.

Let us assume that $X'$ is a $\mu$-variable and infer a contradiction. By the definition of the translation, $X'$ is either of the following: (1) $X$ is a $\mu$-variable and $X' = X_a$, where $a \in$ AVL, or (2) $X$ is a $\nu$-variable and (2-a) $X' = X_{\mathrm{Pos}}$, (2-b) $X' = X_{\mathrm{Fin}}$, or (2-c) $X' = X_{\mathrm{neg}}$. Case (1) is not possible by Lemma 12 (2). Cases (2-a) is eliminated by Lemma 12 (3). Lemma 12 (5) excludes case (2-b). Finally, case (2-c) is impossible by Lemma 12 (4). ∎

**Proof of Lemma 8** The direction from left to right follows from Theorem 1 and Lemma 14. Then, the other direction follows from Lemma 6 (2). ∎

This completes the proof of Theorem 7.

## 5. Conclusions

We proved the decidable and undecidable results on the modal $\mu$-calculus with $\mathbf{N}_\infty$-semantics. The logic is decidable if it does not contain the implication operator. We prove this result by defining a translation $\mathrm{tr}(\varphi)$ of formula $\varphi$ such that the satisfiability of $\varphi$ in $\mathbf{N}_\infty$-semantics is equivalent to the satisfiability of $\mathrm{tr}(\varphi)$ in ordinary semantics. On the other hand, the satisfiability problem becomes undecidable if the logic contains the implication operator.

In future, we plan to strengthen our decidability result to the problem in the form of $[\![\varphi]\!]^{\mathcal{K}}(t) = n$ for given formula $\varphi$ and $n \in \mathbf{N}_\infty$. It may be difficult to extend the translation to this case, because we need to handle complicated conditions. Applying the standard technique of alternating automata[7] is another possibility.

Another direction is the study of the game expression of $\mathbf{N}_\infty$ semantics. We have tried several versions that extend the $\mu$-calculus game for the ordinary semantics, but they were not sufficiently strong for a proof of the correctness of our translation. Therefore, we explicitly used the ordinal numbers, making the proof less intuitive. An appropriate formulation of the semantics using the game terminology is desirable.

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## Appendix

In the appendix, proofs that were omitted in the previous sections are given.

### A.1 Translation

**Lemma 15** The translation process always terminates.

**Proof** We fix a closed formula $\varphi$ of $\mathcal{L}$. Let $A = \mathrm{SF}(\varphi) \times \mathcal{V}$. Let us denote the set of propositional variables that occur in $\varphi$ by $\mathrm{PV}_\varphi$. For $V \in \mathcal{V}$ and $X \in \mathrm{PV}_\varphi$, $\{a \in \mathrm{AV} \mid V(X, a) \neq 0\}$ is denoted by $F(V, X)$.

We define a relation $R$ on $A$ as follows: $((\psi', V'), (\psi, V)) \in R$ if and only if either of the following holds:

- There is $X \in \mathrm{PV}_\varphi$ such that $F(V', X) \supsetneq F(V, X)$ and for any $Y \in \mathrm{PV}_\varphi$ such that $Y \succ X$, $F(V', Y) = F(V, Y)$.
- For any $X \in \mathrm{PV}_\varphi$, $F(V', X) = F(V, X)$ and $\psi' < \psi$.

It is clear that the relation $R$ is well-founded since $\mathrm{PV}_\varphi$, AV, and $\mathrm{SF}(\varphi)$ are all finite.

In all defining equations of the translation in Figure 4, if $\mathrm{tr}(\psi, a, V)$ appears in the left hand side and $\mathrm{tr}(\psi', a', V')$ appears in the right hand side, we have $((\psi', V'), (\psi, V)) \in R$ except for one equation, $\mathrm{tr}(X, a, V) = \mathrm{tr}(\mathrm{BF}(X), a, V)$. For this equation, however, when we go one step further, i.e., when we replace the right hand side $\mathrm{tr}(\mathrm{BF}(X), a, V)$ with its definition, for any occurrence $\mathrm{tr}(\psi', a', V')$ in the right hand side, we have $((\psi', V'), (X, V)) \in R$ as well. Therefore, the translation process terminates. ∎

### A.2 Intermediate Interpretation

**Proof of Lemma 11**

(1) We prove this by induction on the construction of $\varphi$. The only nontrivial part is $\mu X\varphi$ and $\nu X\varphi$. We show only the former, the latter can be proved in a similar way.

Take the series of functions $(F_\alpha)_\alpha$ as in the definition of $[\![\mu X\varphi]\!]^\rho$ (Figure 1). By induction on $\alpha$, we can show that $F_\alpha = \langle X\rangle_{\xi:\alpha}$ and $F_{\alpha+1} = \langle\varphi\rangle_{\xi:\alpha}$. Therefore, $[\![\mu X\varphi]\!]^\rho(t) = \inf\{F_\alpha(t) \mid \alpha \in \mathrm{On}\} = \inf\{\langle\varphi\rangle_{\xi:\alpha}(t) \mid \alpha \in \mathrm{On}\} = \langle\mu X\varphi\rangle_\xi(t)$.

(2) directly follows from (1).

(3) It follows from the fact that there is $\kappa' \in \mathrm{On}$ such that for any $\kappa \geq \kappa'$, $t \in T$, $\varphi \in \varphi_{\mathrm{I}}$, and valuation $\rho$, $[\![\varphi]\!]^\rho(t) = F_{\kappa'}(t)$, together with the proof of (1). ∎

### A.3 $W$-sequence

We call index $l < \mathrm{fv}(\varphi)$ is increasing if either $X = X(\varphi, l)$ is positive in $\varphi$ and $\lambda_X = \nu$ or $X$ is negative in $\varphi$ and $\lambda_X = \mu$; otherwise we call $l$ decreasing.

**Lemma 16** Assume $t \in T$, $\varphi \in \mathrm{SF}(\varphi_{\mathrm{I}})$, and $\xi, \eta \in \mathrm{Seq}_\varphi$ such that either $\xi(l) \leq \eta(l)$ or $\xi(l) \geq \kappa$ and $\eta(l) \geq \kappa$ for any increasing index $l$ , and either $\xi(l) \geq \eta(l)$ or $\xi(l) \geq \kappa$ and $\eta(l) \geq \kappa$ for any decreasing index $l$. Then, we have $\langle\varphi\rangle_\xi(t) \leq \langle\varphi\rangle_\eta(t)$.

**Proof** Can be proved by induction on the lexicographical order on Seq. (If $\xi$ is an extension of $\eta$, we consider $xi$ is smaller than $\eta$.) ∎

**Lemma 17** Assume $w = (\varphi, \xi, t, a) \in W$.

(1) $\mathrm{Suc}(w) \neq \varnothing$.

(2) Assume $w' \in \mathrm{Suc}(w)$. If $\varphi$ is in the form of $\neg\psi$, then $a \in \mathrm{AVL} \iff w'.a \in \mathrm{AVG}$. Otherwise, $a \in \mathrm{AVL} \iff w'.a \in \mathrm{AVL}$

(3) If either $\varphi = \psi_1 \vee \psi_2$ and $a \in \mathrm{AVG}$ or $\varphi{=}\psi_1 \wedge \psi_2$ and $a \in \mathrm{AVL}$, then there exist $w'_1, w'_2 \in \mathrm{Suc}(w)$ such that $w'_k.\varphi = \psi_k$ for $k = 1, 2$.

(4) If either $\varphi = \Diamond\psi$ and $a \in \mathrm{AVG}$ or $\varphi = \Box\psi$ and $a \in \mathrm{AVL}$, then for any $t' \in \mathrm{Suc}(t)$ there exists $w' \in \mathrm{Suc}(w)$ such that $w'.t = t'$.

(5) If $\varphi = \Box\psi$ and $a \in \mathrm{AVG}$, then either of the following holds:
   - There exists $w' \in \mathrm{Suc}(w)$ such that $w'.a = a$.
   - $a = \mathrm{Inf}$ and for any finite subset $T'$ of Suc(t), there exists $w' \in \mathrm{Suc}(w)$ such that $w'.t \in \mathrm{Suc}(t) \setminus T'$ and $w'.a = \mathrm{Pos}$.

(6) If $\varphi = \Box\psi$ and $a = \mathrm{Fin}$, then there is a finite subset $T'$ of $\mathrm{Suc}(t)$ such that for all $t' \in \mathrm{Suc}(t) \setminus T'$ there exists $w' \in \mathrm{Suc}(w)$ such that $w'.t = t'$ and $w'.a = \mathrm{Zer}$.

(7) If either $\varphi = X$ and $\lambda_X = \nu$ or $\varphi = \nu X\psi$, and if $a = \mathrm{Inf}$, then for each $a' \in \{\mathrm{Inf}, \mathrm{Pos}\}$, there is $w' \in \mathrm{Suc}(w)$ such that $w'.a = a'$.

(8) Assume $w' = (\varphi', \xi', t', a') \in \mathrm{Suc}(w)$, $a = \mathrm{Inf}$, $l = \mathrm{pni}(\xi, \varphi) \geq 0$, and $\mathrm{SCont}(\varphi, \xi, t)$. Then, $\varphi$ is not in the form of $\neg\psi$. Moreover, either of the following holds:

- $\xi \upharpoonright l = \xi' \upharpoonright l$ and $\xi(l) > \xi'(l)$.
- $a' = \mathrm{Inf}$, $l = \mathrm{pni}(\xi', \varphi')$, and $\mathrm{SCont}(\varphi', \xi', t')$.

**Proof** Each case can be checked by checking the corresponding definitions. ∎

**Lemma 18** Let $((\varphi_i, \xi_i, t_i, a_i))_i$ be a $W$-sequence, $X \in \mathrm{PV}$, and $l = \mathrm{idx}(X)$.

(1) If $X \in (\mathrm{PV}_\mu \cap \mathrm{PV}_\mathrm{G}) \cup (\mathrm{PV}_\nu \cap \mathrm{PV}_\mathrm{L})$, and $X \in \mathrm{FV}_{\varphi_i}$, then then $\xi_i(l) = \kappa$.

(2) If $\varphi_i = X$, then $\xi_i(l) \geq \xi_{i+1}(l)$. Moreover, if either
- $X \in \mathrm{PV}_\mu \cap \mathrm{PV}_\mathrm{L}$
- $X \in \mathrm{PV}_\nu \cap \mathrm{PV}_\mathrm{G}$ and either $\langle X \rangle_{\xi_i}(t_i) < \infty$ or $\xi_i(l)$ is a successor ordinal number.

then, $\xi_i(l) > \xi_{i+1}(l)$.

(3) Assume $i < i'$ and there is no $j$ $(i \leq j < i')$ such that either of the following holds:
- $\varphi_j = Y \in \mathrm{PV}$, $X \preceq Y$.
- There is $l' \leq l$ such that $\xi_{j+1} = \xi_j\{l' \mapsto \alpha\}$ for some $\alpha < \xi_j(l')$.

Then, $\xi_i \upharpoonright (l+1) = \xi_{i'} \upharpoonright (l+1)$.

**Proof**

(1) Clear from the definition of $R^W$.

(2) If $X \in (\mathrm{PV}_\mu \cap \mathrm{PV}_\mathrm{G}) \cup (\mathrm{PV}_\nu \cap \mathrm{PV}_\mathrm{L})$, this follows from (1).

Assume $X \in \mathrm{PV}_\mu \cap \mathrm{PV}_\mathrm{L}$. Note we also have $a_i \in \mathrm{AVL}$. Since $\langle X \rangle_{\xi_i}(t_i) \in \gamma(a_i)$, we have $\infty > \langle X \rangle_{\xi_i}(t_i) = \inf\{\langle \mathrm{BFS}(X) \rangle_{\hat{\xi}:\beta}(t_i) \mid \beta < \xi_i(l)\}$, where $\hat{\xi} = \xi_i \upharpoonright l$. Therefore $\xi_i(l) > 0$ and there is $\beta < \xi_i(l)$ such that $\langle \mathrm{BFS}(X) \rangle_{\hat{\xi}:\beta}(t_i) = \langle X \rangle_{\xi_i}(t_i)$. By the definition of $R^W$, we have $\xi_{i+1}(l) < \xi_i(l)$.

Assume $X \in \mathrm{PV}_\nu \cap \mathrm{PV}_\mathrm{G}$. In the same argument as above, we have $0 < \langle X \rangle_{\xi_i}(t_i) = \sup\{\langle \mathrm{BFS}(X) \rangle_{\hat{\xi}:\beta}(t_i) \mid \beta < \xi_i(l)\}$. If $\langle X \rangle_{\xi_i}(t_i) < \infty$ or $\xi_i(l)$ is a successor, there is $\beta < \xi(l)$ such that $\langle \mathrm{BFS}(X) \rangle_{\hat{\xi}:\beta}(t_i) = \langle X \rangle_{\xi_i}(t_i)$, therefore $\xi_{i+1}(l) < \xi_i(l)$. Otherwise, $\langle \mathrm{BFS}(X) \rangle_{\hat{\xi}:\xi_i(l)}(t_i) \geq \langle X \rangle_{\hat{\xi}:\xi_i(l)}(t_i) = \sup\{\langle \mathrm{BFS}(X) \rangle_{\hat{\xi}:\beta}(t_i) \mid \beta < \xi_i(l)\} = \infty$. Therefore, $\xi_{i+1}(l) \leq \xi_i(l)$.

(3) Clear from the definition of $R^W$. ∎

**Proof of Lemma 12 (1)** Let $i_1$ be an index such that for all $i \geq i_1$, if $\varphi_i = Y \in \mathrm{PV}$ then $Y \preceq X$. We will show that there are only finitely many $i$ such that $\xi_{i+1} = \xi_i\{l' \mapsto \alpha\}$ for some $\alpha < \xi_i(l')$ and some $l' \leq l$. Then the conclusion follows from Lemma 18, since $(\xi_i(l))_i$ becomes a non-increasing sequence of ordinal numbers.

Assume that there are $l' \leq l$ and infinitely many $i$ such that $\xi_{i+1} = \xi_i\{l' \mapsto \alpha\}$ for some $\alpha < \xi_i(l')$. Let $l'$ be the least such number. Then, again by Lemma 18, there is $i_2 \in \mathbf{N}$ such that $(\xi_i(l') \mid i_2 \leq i \in \mathbf{N})$ is a non-increasing sequence of ordinal numbers and there are infinitely many $i$ such that $\xi_i(l') > \xi_{i+1}(l')$, which is impossible. ∎

**Proof of Lemma 12 (2) and (3)** By (1), there is $i_0 \in \mathbf{N}$ such that for all $i \geq i_0$, $\xi_i \upharpoonright (l+1) = \xi_{i_0} \upharpoonright (l+1)$.

(2) Take $i \geq i_0$ and $\varphi_i = X$. If $X \in \mathrm{PV}_\mu \cap \mathrm{PV}_\mathrm{L}$, by Lemma 18 (2), $\xi_i(l) > \xi_{i+1}(l)$. A contradiction.

(3) Assume $\lambda_X = \nu$ and $X$ is an AVG variable.

Take $i \geq i_0$ such that $\varphi_i = X$. It is sufficient to show that each of $a_{i+1} = \mathrm{Pos}$, $\alpha$ is a successor, and $\langle X \rangle_{\hat{\xi}:\beta}(t_i) = \infty$ implies $\xi_{i+1}(l) < \xi_i(l)$.

First, assume $a_{i+1} = \mathrm{Pos}$. Let $\hat{\xi} = \xi_i \upharpoonright l$. Since $a_i \in \mathrm{AVG}$, we have $\sup\{\langle \varphi_{i+1} \rangle_{\hat{\xi}:\alpha}(t_i) \mid \alpha < \xi_i(l)\} = \langle X \rangle_{\xi_i}(t_i) \in \gamma(a_i)$. Therefore there is $\alpha < \xi_i(l)$ such that $\langle \varphi_{i+1} \rangle_{\hat{\xi}:\alpha}(t_i) > 0$ regardless $a_i = \mathrm{Pos}$ or $a_i = \mathrm{Inf}$. Therefore $\xi_{i+1}(l) < \xi_i(l)$.

Second, if $\xi_i(l)$ is a successor, $\xi_{i+1}(l) < \xi_i(l)$ by Lemma 18 (2).

Third, assume $\langle X \rangle_{\hat{\xi}:\beta}(t_i) = \infty$ for some $\beta < \alpha$. By the argument in the above two paragraphs, we can assume $\alpha$ is limit and $a_i = a_{i+1} = \mathrm{Inf}$. Therefore $\langle \varphi_{i+1} \rangle_{\hat{\xi}:\beta} = \langle X \rangle_{\hat{\xi}:\beta+1} = \langle X \rangle_{\hat{\xi}:\alpha} = \infty$. Hence $\xi_{i+1}(l) \leq \beta < \alpha = \xi_i(l)$. ∎

**Proof of Lemma 12 (4)** By (1) and (3), we can take $i_0 \in \mathbf{N}$, a limit ordinal number $\alpha$, and $\hat{\xi} \in \mathrm{Seq}_{\mathrm{BF}(X)}$ such that $\varphi_{i_0} = X$ and for all $i \geq i_0$,
- $\varphi_i = Y \in \mathrm{PV} \implies Y \preceq X$.
- $\xi_i \upharpoonright (l+1) = \hat{\xi} : \alpha$.
- If $\varphi_i = X$, then $a_i = a_{i+1} = \mathrm{Inf}$ and $\langle X \rangle_{\hat{\xi}:\beta}(t_i) < \infty$ for all $\beta < \alpha$.

We show the following by induction on $i \geq i_0$.

(a) $a_i = \mathrm{Inf}$.

(b) $l = \mathrm{pni}(\xi_i, \varphi_i)$ and $\mathrm{SCont}(\varphi_i, \xi_i, t_i)$.

(c) $\varphi_i \neq \neg\varphi_{i+1}$

First note that (c) follows from (b): if $\varphi_i = \neg\varphi_{i+1}$, then $\infty > \langle \varphi_i \rangle_{\xi_i\{l \mapsto \beta\}}(t_i) = \langle \neg\varphi_i \rangle_{\xi_i\{l \mapsto \beta\}}(t_i)$, therefore $\langle \varphi_i \rangle_{\xi_i\{l \mapsto \beta\}}(t_i) = 0$. Hence $\sup\{\langle \varphi_i \rangle_{\xi_i\{l \mapsto \beta\}}(t_i) \mid \beta <$

$\alpha\} = 0$.

Initial case $i = i_0$ is clear.

Case $\varphi_i = \varphi_{i+1} \vee \psi$ or $\varphi_i = \psi \vee \varphi_{i+1}$. Since $l \in \mathrm{NuLim}(\xi_i, \varphi_i)$ and $\xi_i \upharpoonright (l+1) = \xi_{i+1} \upharpoonright (l+1)$, $l = \mathrm{pni}(\xi_{i+1}, \varphi_{i+1})$ and $\min\{\beta \mid \langle \varphi_{i+1} \rangle_{\xi\{l \mapsto \beta\}}(t_i) = \infty\} = \alpha$. Therefore $\xi_{i+1} = \xi_i$ and $\langle \varphi_{i+1} \rangle_{\xi_{i+1}\{l \mapsto \beta\}}(t_{i+1}) < \infty$ for $\beta < \alpha$. On the other hand, $\sup\{\langle \varphi_{i+1} \rangle_{\xi_{i+1}\{l \mapsto \beta\}}(t_{i+1}) \mid \beta < \alpha\} \geq \sup\{\langle \varphi_i \rangle_{\xi_i\{l \mapsto \beta\}}(t_i) \mid \beta < \alpha\} = \infty$

Case $\varphi_i = \Diamond \varphi_{i+1}$ can be shown in a similar argument.

Case $\varphi_i = \varphi_{i+1} \wedge \psi$ or $\varphi_i = \psi \wedge \varphi_{i+1}$ easily follows from the definition of $R^W$.

Case $\varphi_i = \Box \varphi_{i+1}$. (a): Since $\xi_{i+1}(l) = \xi_i(l)$, $a_{i+1}$ cannot be Pos. (b) easily follows from the definition of $R^W$.

Case $\varphi_i = Y \in \mathrm{PV}$. If $Y = X$, the conclusion follows immediately. Assume $Y \prec X$. If $a_{i+1} = \mathrm{Pos}$, then $\xi_{i+1} = \hat{\xi}\{l \mapsto \alpha\}$, where $\hat{\xi} = \xi_i \upharpoonright \mathrm{idx}(Y)$ and $\alpha$ is the least $\alpha$ that satisfies $\langle \varphi_{i+1} \rangle_{\hat{\xi}\{l \mapsto \alpha\}}(t_i) > 0$. Since $\langle \varphi_{i+1} \rangle_{\hat{\xi}\{l \mapsto \xi(l)\}}(t_i) = \infty$, $\xi_{i+1}(l) < \xi_i(l)$, which is impossible. Therefore $a_{i+1} = \mathrm{Inf}$. Then, $\xi_{i+1} = \hat{\xi} : \beta$ for some $\beta$, and (b) for $i+1$ can be checked easily.

Case $\varphi_i = \lambda Y \psi$. In this case, $Y \prec X$ holds. A similar argument to the above case can be applied. ∎

**Proof of Lemma 12 (5)** Let $i_0 \in \mathbf{N}$ such that $\varphi_{i_0} = X$ and for all $i \geq i_0$, $\varphi_i \neq \neg \varphi_{i+1}$, and if $\varphi_i = Y \in \mathrm{PV}$ then $Y \preceq X$. Let $i_0, i_1, \ldots$ be the enumeration of the indices $i \geq i_0$ such that $\varphi_i = X$. Let $l = \mathrm{idx}(X)$.

Let $n_i = \langle \varphi_i \rangle_{\xi_i}(t_i)$. By checking the definition of the relation $R^W$, we can show that $a_i \in \mathrm{AVL}$ and $n_i \geq n_{i+1}$ for all $i \geq i_0$. Therefore there is $K \in \mathbf{N}$ and $c \geq 0$ such that $n_i = c$ for all $i \geq i_K$.

Note that $\xi_i(l) = \kappa$ for any $i \geq i_K$ since $\lambda_X = \nu$ and $a_i \in \mathrm{AVL}$. Let $\alpha$ be an ordinal number. For any $i, j \geq i_K$, $\xi_i\{l \mapsto \alpha\}$ is an extension of $\xi_j\{l \mapsto \alpha\}$ or $\xi_j\{l \mapsto \alpha\}$ is an extension of $\xi_i\{l \mapsto \alpha\}$. Also, we have $\langle \varphi_i \rangle_{\xi_i}(t_i) \geq \langle \varphi_i \rangle_{\xi_i\{l \mapsto \alpha\}}(t_i)$ for any $i \geq i_K$ by Lemma 16.

We claim that $\langle \varphi_i \rangle_{\xi_i\{l \mapsto \alpha\}}(t_i) = 0$ for all $i \geq i_K$ and for all ordinal number $\alpha$.

First, let $\alpha = 0$. By definition of $\langle \cdot \rangle$., we have $\langle \varphi_{i_k} \rangle_{\xi_{i_k}\{l \mapsto 0\}}(t_{i_k}) = 0$. Then, we can use induction on $i$ with reverse direction (case $i$ depends on case $i+1$) to show $\langle \varphi_i \rangle_{\xi_i\{l \mapsto 0\}}(t_i)$ for $i \geq i_K$.

- Case $\varphi_i = \psi_0 \vee \psi_1$. Either $\psi_0$ or $\psi_1$ is $\varphi_{i+1}$. Since $\langle \varphi_{i+1} \rangle_{\xi_{i+1}\{l \mapsto 0\}}(t_{i+1}) = 0$ by the induction hypothesis, we have $\langle \varphi_i \rangle_{\xi_i\{l \mapsto 0\}}(t_i) = \min\{\langle \psi_j \rangle_{\xi_i\{l \mapsto 0\}}(t_i) \mid j = 0, 1\} = 0$.
- Case $\varphi_i = \psi_0 \wedge \psi_1$. Let $\psi_j = \varphi_{i+1}$ and $\psi = \psi_{1-j}$. We have $\langle \psi \rangle_{\xi_i}(t_i) = 0$, since $\infty > c = \langle \varphi_i \rangle_{\xi_i}(t_i) = \langle \varphi_{i+1} \rangle_{\xi_i}(t_i) + \langle \psi \rangle_{\xi_i}(t_i) = c + \langle \psi \rangle_{\xi_i}(t_i)$. Therefore, $\langle \psi \rangle_{\xi_i\{l \mapsto 0\}}(t_i) = 0$. Since $\langle \varphi_{i+1} \rangle_{\xi_i\{l \mapsto 0\}}(t_i) = 0$ by the induction hypothesis, we have $\langle \varphi_i \rangle_{\xi_i\{l \mapsto 0\}}(t_i) = 0 + 0 = 0$.
- Case $\varphi_i = \Diamond \psi$ and $\varphi_i = \Box \psi$. A similar argument as in the previous cases can be applied.
- Case $\varphi_i = Y \in \mathrm{PV}$. Note that $Y \prec X$ since $i \geq i_K$. Since $\varphi_{i+1} = \mathrm{BFS}(Y)$, the induction hypothesis is $\langle \mathrm{BFS}(Y) \rangle_{\xi_{i+1}\{l \mapsto 0\}} = 0$. By Lemma 11 (3), we have $\langle Y \rangle_{\xi_i\{l \mapsto 0\}} = \langle \mathrm{BFS}(Y) \rangle_{\xi_{i+1}\{l \mapsto 0\}}$
- Case $\varphi_i = \lambda Y \psi$. Note that $Y \prec X$ since $i \geq i_K$. Since $\varphi_{i+1} = \mathrm{BFS}(\psi)$, the induction hypothesis is $\langle \psi \rangle_{\xi_{i+1}\{l \mapsto 0\}} = 0$. By Lemma 11 (3), we have $\langle \lambda Y \psi \rangle_{\xi_i\{l \mapsto 0\}} = \langle \psi \rangle_{\xi_{i+1}\{l \mapsto 0\}}$.

Next, let $\alpha = \beta + 1$. Since $\varphi_{i_k} = X$, $\varphi_{i_k+1} = \mathrm{BFS}(X)$, $\lambda_X = \nu$, $\xi_{i_k}\{l \mapsto \beta + 1\} = (\xi_{i_k} \upharpoonright l) : (\beta + 1) : \kappa : \cdots$, and $\xi_{i_k+1}\{l \mapsto \beta\} = (\xi_{i_k} \upharpoonright l) : \beta$, we have $\langle \varphi_{i_k} \rangle_{\xi_{i_k}\{l \mapsto \beta+1\}}(t_{i_k}) = 0$ using the induction hypothesis on $\alpha$, namely, $\langle \varphi_{i_k+1} \rangle_{\xi_{i_k+1}\{l \mapsto \beta\}}(t_{i_k+1}) = 0$. Then, $\langle \varphi_i \rangle_{\xi_i\{l \mapsto \beta+1\}}(t_i) = 0$ for $i \geq i_K$ can be shown by the same argument as in the case $\alpha = 0$. Finally, let $\alpha$ be a limit ordinal number. $\langle \varphi_{i_k} \rangle_{\xi_{i_k}\{l \mapsto \alpha\}}(t_{i_k}) = 0$ follows from the induction hypothesis on $\alpha$, and for other $i \geq i_K$, $\langle \varphi_i \rangle_{\xi_i\{l \mapsto \alpha\}}(t_i) = 0$ can be checked in the same argument. This establishes the claim.

By taking $i = i_K$ and $\alpha = \kappa$, we have $c = \langle \varphi_{i_K} \rangle_{\xi_{i_K}\{l \mapsto \kappa\}}(t_{i_K}) = 0$, since $\xi_{i_K}\{l \mapsto \kappa\} = \xi_{i_K}$. ∎

### A.4 $(\varphi, a)$-simulation

We prove Lemmas 9 and 10 using a series of lemmas.

**Lemma 19** Assume $\mathcal{K} = (T, R, L)$ is a Kripke structure for $\mathcal{L}$, $t_{\mathrm{I}} \in T$, $a_{\mathrm{I}} \in \mathrm{AV}$, and $[\![\varphi_{\mathrm{I}}]\!]^{\mathcal{K}}(t_{\mathrm{I}}) \in \gamma(a_{\mathrm{I}})$. Then, there is a tree-shape Kripke structure $\mathcal{K}_1 = (T_1, R_1, L_1)$ for $\mathcal{L}$ such that $[\![\varphi_{\mathrm{I}}]\!]^{\mathcal{K}_1}(t_0) \in \gamma(a_{\mathrm{I}})$, where $t_0$ is the root of the tree $\mathcal{K}_1$.

**Proof** This lemma be shown by an ordinary unwinding argument. ∎

For $t \in T$ and $w \in W$ such that $w.t = t$, we call $w' \in W$ a $t$-descendant of $w$

if there is a finite $W$-sequence $(w_i \mid i \le i \le i_0)$ such that $w_0 = w$, $w_{i_0} = w'$ and $w_i.t = t$ for $0 \le i \le i_0$.

**Lemma 20** Assume that $\mathcal{K} = (T, R, L)$ be a Kripke structure for $\mathcal{L}$, $t \in T$, and $w = (\varphi_0, \xi_0, t, a_0) \in W$. Then, there exist only finitely many $t$-descendants of $w$.

**Proof** We construct a tree so that an element of $W$ is assigned to each node of the tree.

Initial step: $(\varphi_0, \xi_0, t, a_0)$ is assigned to the root node. The root node is open.

Succeeding steps: for each open leaf node, let $w$ be the assigned element of $W$ and let $Z = \{w' \in W \mid (w, w') \in R^W\}$. For each $w' \in Z$, we extend the node with a new leaf node, which $w'$ is assigned. If $w'$ appear in one of its ancestors, the node is closed, otherwise it is open.

Since every element of $W$ that satisfies the condition of this lemma appears as an assigned element of the tree, what we need to show is that the tree is finite. By checking the definition of $R^W$, it can be easily shown that the tree is finitely branching. By König's lemma, it is sufficient to show that there is no infinite branch.

Suppose that there is an infinite branch of the tree. Let $(\varphi_i, \xi_i, t, a_i)$ be the assigned element of $W$ to the $i$-th level. Since this is a $W$-sequence, by Lemma 12 (1), there is $i_0 \in \mathbf{N}$ such that for any $i \ge i_0$, $\xi_i \restriction (l+1) = \xi_{i_0} \restriction (l+1)$, where $l = \mathrm{idx}(X)$ and $X$ is the principal variable of the $W$-sequence. Let $i_0, i_1, \dots$ be the enumeration of $i$ such that $i \ge i_0$ and $\varphi_i = X$. Then, for all $k \in \mathbf{N}$, $\xi_{i_k+1} = \xi_{i_0} \restriction (l+1)$ since the length of $\xi_{i_k+1}$ is $l+1$. Since there are only finitely many $\varphi$ and $a$ that can appear in a $W$-sequence, that means there is some $i < j$ such that $(\varphi_i, \xi_i, t, a_i) = (\varphi_j, \xi_j, t, a_j)$, therefore the branch must be closed at $(\varphi_j, \xi_j, t, a_j)$, a contradiction. ∎

**Lemma 21** For any tree-shape Kripke structure $\mathcal{K} = (T, R, L)$ for $\mathcal{L}$ (we denote its root by $t_\mathrm{I}$) and $a_\mathrm{I} \in \mathrm{AV}$, there exist a Kripke structure $\mathcal{K}' = (S, R', L')$ for $\mathcal{L}'$, $s_\mathrm{I} \in S$, and a $(\varphi_\mathrm{I}, a_\mathrm{I})$-simulation $Q$ between $(\mathcal{K}, t_\mathrm{I})$ and $(\mathcal{K}', s_\mathrm{I})$.

**Proof** We construct $\mathcal{K}'$ in the shape of a tree. A function $h : S \to T$ is simultaneously constructed and we define $Q = \{(h(s), s) \mid t \in T\}$. Let $\overline{W}$ be the set of descendants of $w_\mathrm{I} = (\varphi_\mathrm{I}, \epsilon, t_\mathrm{I}, a_\mathrm{I})$. A finite subset $Z(s)$ of $\overline{W}$ is assigned to each node $s$ of the tree. For all $w \in Z(s)$, $w.t = h(s)$.

Initial step: we start with the root node $s_\mathrm{I}$. We define $h(s_\mathrm{I}) = t_\mathrm{I}$ and $Z(s_\mathrm{I}) = \{w_\mathrm{I}\}$.

Succeeding steps: let $s$ be a leaf node, $Z = Z(s)$, and $t = h(s)$. Let

- $\Phi_\Diamond$ be the set of $\mathrm{SF}(\varphi)$ in the form of $\Diamond\psi$,
- $\Phi_\Box$ be the set of $\mathrm{SF}(\varphi)$ in the form of $\Box\psi$,
- $Z'$ be the set of $t$-descendants of some $w \in Z$,
- $Z_\mathrm{D} = \{w \in Z' \mid$ either $w.\varphi \in \Phi_\Diamond$ and $w.a \in \mathrm{AVL}$ or $w.\varphi \in \Phi_\Box$ and $w.a \in \mathrm{AVG}\}$, and
- $Z_\mathrm{B} = \{w \in Z' \mid$ either $w.\varphi \in \Phi_\Diamond$ and $w.a \in \mathrm{AVG}$ or $w.\varphi \in \Phi_\Box$ and $w.a \in \mathrm{AVL}\}$.

If $\varphi = \Diamond\psi$ or $\varphi = \Box\psi$, we denote $\psi$ by $\vec{\varphi}$. By Lemma 20, $Z_\mathrm{D}$ and $Z_\mathrm{B}$ are finite.

For each $w_1 = (\varphi_1, \xi_1, t, a_1) \in Z_\mathrm{D}$, we create a new leaf node $s'$ and extend $s$ with $s'$. Let $\psi_1 = \overrightarrow{\varphi_1}$.

(a) If there exists $\overline{w_1} \in \mathrm{Suc}(w_1)$ such that $\overline{w_1}.a = a_1$, we put $(s, s')$ in $R'(1)$. Let $t' = \overline{w_1}.t$. By Lemma 17, for each $w \in Z_\mathrm{B}$, there exists $\overline{w} \in \mathrm{Suc}(w)$ such that $\overline{w}.t = t'$ and $\overline{w}.a = a$.

(b) Otherwise, we put $(s, s')$ in $R'(\infty)$. Let $Z'_\mathrm{B} = \{w \in Z_\mathrm{B} \mid w.a = \mathrm{Fin}\}$. For each $w \in Z'_\mathrm{B}$, let $T(w)$ be the finite subset of $\mathrm{Suc}(t)$ guaranteed by Lemma 17, i.e., for all $t' \in \mathrm{Suc}(t) \setminus T(w)$, there exists $\overline{w} \in \mathrm{Suc}(w)$ such that $\overline{w}.t = t'$ and $\overline{w}.a = \mathrm{Zer}$. Let $T' = \bigcup_{w \in Z'_\mathrm{B}} T(w)$. Since $Z'_\mathrm{B}$ is finite, so is $T'$. Therefore, by Lemma 17, there exists $\overline{w_1} \in \mathrm{Suc}(w_1)$ such that $\overline{w_1}.t \in \mathrm{Suc}(t) \setminus T'$ and $\overline{w_1}.a = \mathrm{Pos}$. Let $t' = \overline{w_1}.t$. For $w \in Z_\mathrm{B} \setminus Z'_\mathrm{B}$, take $\overline{w} \in \mathrm{Suc}(w)$ such that $\overline{w}.t = t'$ and $\overline{w}.a = a$, guaranteed by Lemma 17.

In both cases, We define $h(s') = t'$, and $Z(s') = \{\overline{w} \mid w \in \{w_1\} \cup Z_\mathrm{B}\}$.

This completes the construction. The labeling function is defined by $L'(s) = \{p_0\}$ if $L(p, h(s)) = 0$, $L'(s) = \{p_\infty\}$ if $L(p, h(s)) = \infty$, and $L'(s) = \varnothing$ if $0 < L(p, h(s)) < \infty$.

Conditions from (1) to (3) of $(\varphi_\mathrm{I}, a_\mathrm{I})$-simulation can be checked easily.

It can be easily shown that if $(t, s) \in Q$, $w' = (\varphi, \xi, t, a) \in W$ is a descendant of $w_\mathrm{I}$, and $\varphi \in \Phi_\Diamond \cup \Phi_\Box$, then $w' \in Z'$. Note that $h$ is one-to-one because $\mathcal{K}$ is a tree.

Using this fact, condition (4) can also be checked. ∎

**Proof of Lemma 9** Clear from Lemmas 19 and 21. ∎

**Lemma 22** For any closed satisfiable formula $\chi$ in $\mathcal{L}'$, there is a Kripke structure $\mathcal{K}' = (S, R', L')$ in the shape of a finitely branching tree that satisfies $\chi$ at its root. I.e.,

- $(S, R'(1) \cup R'(\infty))$ forms a tree.
- $\mathcal{K}', s_{\mathrm{I}} \models \chi$, where $s_{\mathrm{I}}$ is the root of the tree.
- For any $s$, $\{s' \in S \mid (s, s') \in R'(1) \cup R'(\infty)\}$ is finite.
- $R'(1) \cap R'(\infty) = \varnothing$.

**Proof** A tree-shape Kripke structure $\mathcal{K}'' = (S'', R'', L'')$ that satisfies $\chi$ can be built by an ordinary unwinding argument. Let $\sigma$ be a memoryless winning strategy of Player for $\chi$. Strategy $\sigma$ can be formulated as a partial function on $\mathrm{SF}(\chi) \times S''$, and contains both information for Player and Opponent. Remove all transitions $(s, s') \in R''$ such that there is no $\varphi \in \mathrm{SF}(\chi)$ such that $(\varphi, s) \in \mathrm{dom}(\sigma)$ and $\sigma(\varphi, s) = s'$. This does not alter the winning regions of Player and Opponent, and for each $s \in S''$, only finitely many $s'$ remains since $\mathrm{SF}(\chi)$ is finite. ∎

**Lemma 23** For any Kripke structure $\mathcal{K}' = (S, R', L')$ for $\mathcal{L}'$ in the shape of a finitely branching tree, $s_{\mathrm{I}} \in S$, and $a_{\mathrm{I}} \in \mathrm{AV}$, there exist a Kripke structure $\mathcal{K} = (T, R, L)$ for $\mathcal{L}$, $t_{\mathrm{I}} \in T$, and a $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-simulation $Q$ between $(\mathcal{K}, t_{\mathrm{I}})$ and $(\mathcal{K}', s_{\mathrm{I}})$.

**Proof** For $m \in \{1, \infty\}$, let $S_m = \{s \in S \mid \text{there exists } \hat{s} \text{ such that } (\hat{s}, s) \in R(m)\}$. Then, $\{\{s_{\mathrm{I}}\}, S_1, S_\infty\}$ is a partition of $S$. Also, for $i \in \mathbf{N}$, let $S^i$ be the set of elements of $S$ whose depth is $i$: $S^0 = \{s_{\mathrm{I}}\}$, $S^{i+1} = \{s \in S \mid \text{there is } \hat{s} \in S^i$ such that $(\hat{s}, s) \in R'(1) \cup R'(\infty)\}$.

We construct $T$ from $S$ by adding infinitely many copies of elements of $S_\infty$. Formally, let $S_{\mathrm{D}} = \{(s, n) \in S \times \mathbf{N} \mid n > 0 \implies s \in S_\omega\}$. For $d = (s, n) \in S_{\mathrm{D}}$, we write $s = d.S$ and $n = d.\mathbf{N}$. Then we define $T = \{t : n \to S_{\mathrm{D}} \mid 1 \leq n \in \mathbf{N}$, $t(i).S \in S^i$, $(t(i).S, t(i+1).S) \in R'(1) \cup R'(\infty)$ for $i < n\}$. The relation is $(t, t') \in R \iff \mathrm{dom}(t') = \mathrm{dom}(t) + 1$ and $t' \restriction \mathrm{dom}(t) = t$, therefore $\mathcal{K}$ is also in the form of (infinitely-branching) tree. $t_{\mathrm{I}}$ is the root of the tree: $\mathrm{dom}(t_{\mathrm{I}}) = 1$ and $t_{\mathrm{I}}(0) = (s_{\mathrm{I}}, 0)$. For $t \in T$, the $S$-component of the last element of $t$ is denoted by $s(t)$, i.e., $s(t) = t(\mathrm{dom}(t) - 1).S$. The labeling function is defined by:

$$L(t,p) = \begin{cases} 0 & \text{if } s(t) \in L'(p_0) \setminus L'(p_\infty) \\ \infty & \text{else if } s(t) \in L'(p_\infty) \setminus L'(p_0) \\ 1 & \text{otherwise} \end{cases}$$

$Q$ is defined by $(t, s) \in Q \iff s(t) = s$.

Observe that for any $t_1, t_2 \in T$, $\varphi \in \mathrm{SF}(\varphi_{\mathrm{I}})$, and $\xi \in \mathrm{Seq}_\varphi$, $s(t_1) = s(t_2)$ implies $\langle\varphi\rangle_\xi(t_1) = \langle\varphi\rangle_\xi(t_2)$. This is because Kripke sub-structures $\mathcal{K} \restriction T_1$ and $\mathcal{K} \restriction T_2$ are isomorphic, where $T_k = \{t' \in T \mid (t, t') \in R^*\}$ for $k = 1, 2$.

The first three conditions of $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-simulation are clearly satisfied. Let us check the condition (4).

Assume either $\varphi = \Diamond\psi$ and $a \in \mathrm{AVL}$ or $\varphi = \Box\psi$ and $a \in \mathrm{AVG}$. By Lemma 17 (1), there is $w' \in W$ such that $(w, w') \in R^W$. The conclusion trivially holds if $a = w'.a$, so assume $(a, w'.a) = (\mathrm{Inf}, \mathrm{Pos})$. By the above observation, there must be infinitely many successors of $t$, therefore $(s, s') \in R'(\infty)$.

Next, assume $s' \in S$, $(s, s') \in R'(1) \cup R'(\infty)$, and either $\varphi = \Diamond\psi$ and $a \in \mathrm{AVG}$ or $\varphi = \Box\psi$ and $a \in \mathrm{AVL}$. Then, by Lemma 17 (4), there is $w' \in \mathrm{Suc}(w)$ such that $w'.t = h(s')$. Clearly, $(w'.t, s') \in Q$. Moreover, if $(s, s') \in R'(\infty)$ and $a = \mathrm{Fin}$, again, by the above observation, $w'.a = \mathrm{Zer}$. ∎

**Proof of Lemma 10** Clear from Lemmas 22 and 23. ∎

### A.5 Strategy

**Proof of Lemma 13** We construct $\sigma$, $(w_i)_i$, $(s_i)_i$, and $(V_i)_i$ so that the position of the game becomes $\mathrm{tr}(\varphi_i, a_i, V_i)$, where $w_i = (\varphi_i, \xi_i, t_i, a_i)$.

As the initial step, we define: $w_0 = (\varphi, \epsilon, t_{\mathrm{I}}, a_{\mathrm{I}})$, $s_0 = s_{\mathrm{I}}$, and $V_0 = V_{\mathrm{I}}$.

The succeeding steps are defined as follows, depending on cases. In the following cases other than $\varphi = \Diamond\psi$ or $\varphi = \Box\psi$, we take $s_{i+1} = s_i$.

**Case** $\varphi_i = \neg\psi$

We take $w_{i+1}$ so that $w_{i+1} \in \mathrm{Suc}(w_i)$ by Lemma 17 (1). In this case, the current position of the play does not change: $\mathrm{tr}(\varphi_i, a_i, V_i) = \mathrm{tr}(\varphi_{i+1}, a_{i+1}, V_i)$.

**Case** either $\varphi_i = \psi_0 \vee \psi_1$ and $a_i \in \mathrm{AVL}$ or $\varphi_i = \psi_0 \wedge \psi_1$ and $a_i \in \mathrm{AVG}$

Player's turn. We take $w_{i+1} \in \mathrm{Suc}(w_i)$ by Lemma 17 (1) and Player's move is $(\mathrm{tr}(\varphi_{i+1}, a_{i+1}, V_i), s_{i+1})$.

**Case** either $\varphi_i = \psi_0 \vee \psi_1$ and $a_i \in \mathrm{AVG}$ or $\varphi_i = \psi_0 \wedge \psi_1$ and $a_i \in \mathrm{AVL}$

Opponent's turn.

Let $k \in \{0,1\}$ be the index such that Opponent selects $\mathrm{tr}(\psi_k, a_i, V_i)$. We define $\varphi_i = \psi_k$. By Lemma 17 (3), we take $w_{i+1} \in \mathrm{Suc}(w_i)$ such that $w_{i+1}.\varphi = \psi_k$.

**Case** either $\varphi_i = \Diamond\psi$ and $a_i \in \mathrm{AVL}$ or $\varphi_i = \Box\psi$ and $a_i \in \mathrm{AVG}$

Player's turn. We take $w' \in W$ and $s' \in S$ in the condition (4) of the definition of the $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-simulation, and let $w_{i+1} = w'$ and $s_{i+1} = s'$. Player's two successive moves are $(\langle m\rangle\mathrm{tr}(\varphi_{i+1}, a_{i+1}, V_i), s_i)$ and $(\mathrm{tr}(\varphi_{i+1}, a_{i+1}, V_i), s_{i+1})$, where $m$ is such that $(s_i, s_{i+1}) \in R(m)$.

Note that this also shows that the play does not terminate in this stage.

**Case** either $\varphi_i = \Diamond\psi$ and $a_i \in \mathrm{AVG}$ or $\varphi_i = \Box\psi$ and $a_i \in \mathrm{AVL}$

Opponent's turn.

Let $([m]\mathrm{tr}(\psi, a', V_i), s_i)$ and $(\mathrm{tr}(\psi, a', V_i), s')$ be Opponent's two successive moves ($s' \in S$). We take $s_{i+1} = s'$. Let $m$ be such that $(s_i, s_{i+1}) \in R'(m)$. Since $Q$ is a $(\varphi_{\mathrm{I}}, a_{\mathrm{I}})$-simulation, there exists $w' \in \mathrm{Suc}(w)$ such that $(w'.t, s_{i+1}) \in Q$. If $m = \infty$ and $a = \mathrm{Fin}$, then $w'.a = \mathrm{Zer}$. We take $w_{i+1} = w'$.

**Case** $\varphi_i = \lambda X\psi$ or $\varphi_i = X$

When $\varphi_i = X$, we define $\psi = \mathrm{BFS}(X)$.

(Subcase 1) If $\lambda_X = \mu$ or $a_i \in \{\mathrm{Zer}, \mathrm{Pos}\}$, then $\mathrm{tr}(\varphi_i, a_i, V_i)$ is either $X_{a_i}$ or $\lambda'' X_{a_i}\mathrm{tr}(\psi, a_i, V)$ for some $V \in \mathcal{V}$, where $\lambda''$ is either $\mu$ or $\nu$. In both cases, after finitely many moves of Player, the position of the play becomes $(\mathrm{tr}(\psi, a_i, V), s_i)$ for some $V \in \mathcal{V}$. We take $w_{i+1} \in \mathrm{Suc}(w_i)$ by Lemma 17 (1). By definition of $R^W$, we have $\varphi_{i+1} = \psi$, $a_{i+1} = a_i$.

(Subcase 2) If $\lambda_X = \nu$ and $a_i = \mathrm{Fin}$, then $\mathrm{tr}(\varphi_i, a_i, V_i)$ is either $X_{\mathrm{Fin}}$, $X_{\mathrm{neg}}$, or $\nu X_{\mathrm{neg}}\mu X_{\mathrm{Fin}}(\mathrm{tr}(\psi, \mathrm{Fin}, V) \wedge \mathrm{tr}(\nu X\psi, \mathrm{Zer}, V))$ for some $V \in \mathcal{V}$. In any of the cases, after finitely many moves of Player, the position of the play becomes $(\mathrm{tr}(\psi, \mathrm{Fin}, V) \wedge \mathrm{tr}(\nu X\psi, \mathrm{Zer}, V), s_i)$ for some $V \in \mathcal{V}$. This is Player's turn. By Lemma 17 (1), we take $w_{i+1} \in \mathrm{Suc}(w_i)$. By the definition of $R^W$, $a_{i+1}$ is either Fin or Zer. If $a_{i+1} = \mathrm{Fin}$, Player moves to $(\mathrm{tr}(\psi, \mathrm{Fin}, V), s_i)$, else to $(\mathrm{tr}(\nu X\psi, \mathrm{Zer}, V), s_i)$. In the latter case, again, with finitely many moves of Player, the position of the play becomes $(\mathrm{tr}(\psi, \mathrm{Zer}, V'), s_i)$ for some $V' \in \mathcal{V}$.

(Subcase 3) If $\lambda_X = \nu$ and $a_i = \mathrm{Inf}$, then a similar process to Subcase 2 can be applied. Difference is that after finitely many moves of Player, the position of the play becomes $(\mathrm{tr}(\psi, \mathrm{Inf}, V) \wedge \mathrm{tr}(\nu X\psi, \mathrm{Pos}, V), s_i)$ for some $V \in \mathcal{V}$, which is an Opponent's turn. If Opponent chooses $\mathrm{tr}(\psi, \mathrm{Inf}, V)$, we define $a' = \mathrm{Inf}$, else $a' = \mathrm{Pos}$. By Lemma 17 (7), we take $w_{i+1} \in \mathrm{Suc}(w_i)$ such that $w_{i+1}.a = a'$. The rest of the process can be done in a similar way as in Subcase 2.

**Case** $\varphi_i = p$

If $a_i = \mathrm{Zer}$, the current position of the play is $(p_0 \wedge \neg p_\infty, s_i)$, Opponent's turn. Since $0 = \langle p\rangle_{\xi_i}(t_i) = L(p, t_i)$ and $(t_i, s_i) \in Q$, we have $s_i \in L'(p_0) \setminus L'(p_\infty)$. Therefore regardless Opponent chooses either move, Player wins.

If $a_i = \mathrm{Pos}$, the current position is $(\neg p_0 \vee p_\infty, s_i)$, Player's turn. Since $0 < \langle p\rangle_{\xi_i}(t_i) = L(p, t_i)$ and $(t_i, s_i) \in Q$, we have either $s_i \in L'(p_\infty)$ or $s_i \notin L'(p_0)$. Player's move is $(p_\infty, s_i)$ in the first case, and $(\neg p_0, s_i)$ in the second. Thus, Player wins.

A similar argument shows that Player wins in remaining two cases.

**Case** $\varphi_i = 1$

Since $1 = \langle 1\rangle_{\xi_i}(t_i) \in \gamma(a_i)$, $a_i$ is either Fin or Pos. Therefore, the current position is $(\mathbf{true}, s_i)$, namely, Player wins.

This completes the construction of $\sigma$ and $w_i$. Now, it should be easy to confirm that two conditions stated are satisfied. ∎